JOINTLY QUASINORMAL ISOMETRIES
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ABSTRACT. If $U$ and $V$ are isometries each of which commutes with $U^*V$ and $V^*U$, then a necessary and sufficient condition that $U$ and $V$ commute is that the ranges of $U$ and $V$ are equal. This result leads to the construction of a subnormal-valued analytic function which has no normal extension.

In [2] Globevnik and Vidav proved that if $f$ is an analytic function whose values are normal operators on a Hilbert space $X$, then the range of $f$ is abelian. In [1] Fleming and Jamison ask if this result is valid when the values of a function are subnormal or even quasinormal. A related question is whether an analytic subnormal-valued function has an extension to an analytic normal-valued function. The answer to each of these questions is no, as will be seen in Example 2.

A sufficient condition that the values of an analytic function $f$ be quasinormal is that $A(B^*C) = (B^*C)A$ whenever $A$, $B$, and $C$ are coefficients of $f$. If this condition holds and $A$ and $B$ are coefficients of $f$, then each of $A$ and $B$ commutes with each of $A^*A$, $A^*B$, $B^*A$ and $B^*B$, in which case we shall call $A$ and $B$ jointly quasinormal. For the simple analytic function $f(z) = A + zB$ we can now paraphrase the question in [1] by asking whether $A$ and $B$ commute when $A$ and $B$ are jointly quasinormal. The answer to this question is also no, as will be seen in Example 1.

The key to the answers of the above-mentioned questions is in the following theorem concerning isometries. The terminology used in the paper is as follows: $A$ is normal if $A$ commutes with $A^*$, quasinormal if $A$ commutes with $A^*A$, an isometry if $A^*A = I$, and a partial isometry if $A^*A$ is a projection. Basic facts concerning these special operators can be found in [3]. The range of an operator $A$ is denoted by $A(X)$.

THEOREM. If $U$ and $V$ are jointly quasinormal isometries, the following are equivalent:

(i) $UV = VU$,
(ii) $UV(X) = VU(X)$,
(iii) $U(X) = V(X)$.

PROOF. (i)$\Rightarrow$(ii) trivially. To see that (ii)$\Rightarrow$(iii) note that if $UV(X) = VU(X)$ then $U^*UV(X) = U^*VU(X)$. Consequently, $V(X) = U(U^*V)(X)$ since $U$ is an isometry and $U$ and $V$ are jointly quasinormal. Thus, $V(X) \subset U(X)$ if $UV(X) = VU(X)$ and by symmetry $U(X) \subset V(X)$ also. To see that (iii)$\Rightarrow$(i) assume that $U(X) = V(X)$ or equivalently $UU^* = VV^*$ since $U$ and $V$ are (partial) isometries. Let $K = UV - VU$ and note that $K(X) \subset U(X)$ since $V(X) \subset U(X)$. Furthermore, $U^*K = U^*UV - U^*VU = V - UU^*V$ (since $U$ is an isometry and $U$ and $V$ are...
jointly quasinormal) = \( V - VV^*V \) (since \( VV^* = UU^* \)) = 0 (since \( V \) is an isometry). Therefore, \( U^*K = 0 \), so that \( K(X) \) is orthogonal to \( U(X) \). We previously showed \( K(X) \subset U(X) \) also. These two results imply that \( K = 0 \) or that \( UV = VU \), as desired. Q.E.D.

This theorem makes the task of constructing noncommuting jointly quasinormal operators easy. Alan Lambert first suggested the simple construction in Example 1.

**EXAMPLE 1.** Let \( X \) be a Hilbert space with orthonormal basis \( \{e_n : n = 1, 2, \ldots \} \). Let \( U \) and \( V \) be the isometries for which \( Ue_n = e_{2n} \) and \( Ve_n = e_{2n-1} \). \( U \) and \( V \) do not commute since \( UVe_1 = e_2 \) and \( VUe_1 = e_3 \). On the other hand, \( U^*V = V^*U = 0 \) since \( U(X) \) and \( V(X) \) are orthogonal. Since \( U^*U = V^*V = I \), all of the commutation properties for the joint quasinormality of \( U \) and \( V \) are satisfied trivially.

**EXAMPLE 2.** Let \( U \) and \( V \) be the noncommuting jointly quasinormal isometries in Example 1 and define \( f(z) = U + zV \) for each complex number \( z \). Note that \( f(z)^*f(z) = (1 + |z|^2)I \) where \( I \) is the identity operator, so that each value \( f(z) \) is quasinormal, and consequently, subnormal. Thus we have an example of a subnormal-valued analytic function with nonabelian range. To see that this also provides us with a subnormal-valued analytic function which does not have a normal-valued analytic extension we need only recall that such an extension would have an abelian range [2]. This, of course, would force \( f \) to have an abelian range.

We close with two observations. If \( A \) and \( B \) are jointly quasinormal operators with canonical polar decompositions \( UP \) and \( VQ \), respectively, then \( U \) and \( V \) are jointly quasinormal partial isometries. (The proof of this depends upon a rather lengthy, but elementary, algebraic computation.) Moreover, the jointly quasinormal operators \( A \) and \( B \) commute exactly when \( U \) and \( V \) commute. Secondly, it follows easily from the Theorem that when \( U \) and \( V \) are jointly quasinormal partial isometries, a necessary and sufficient condition for \( U \) and \( V \) to commute is that \( UV(X) = VU(X) \). Thus, the general question of commutativity of quasinormal operators reduces to consideration of their partially isometric factors.

**REFERENCES**


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