CUBE SLICING IN \( \mathbb{R}^n \)

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ABSTRACT. We prove that every \((n - 1)\)-dimensional section of the unit cube in \( \mathbb{R}^n \) has volume at most \( \sqrt{2} \). This upper bound is clearly best possible.

1. Introduction. Hensley [3] showed that if \( Q_n = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbb{R}^n \) is the central unit cube in \( \mathbb{R}^n \), and \( H \) is an \((n - 1)\)-dimensional subspace of \( \mathbb{R}^n \), then \(|H \cap Q_n|\) lies between 1 and 5. (Here and henceforth \(|\cdot|\) will denote the usual volume in a Euclidean space.) Hensley conjectured that the volume is at most \( \sqrt{2} \). Besides proving this conjecture, we shall provide a somewhat simpler proof that the volume is at least 1.

The bound 5, obtained by Hensley, can be improved in a more general setting. In [1], we show that there is an absolute constant, \( c \), so that for all \( n \), every \( n \)-dimensional Banach space has a representation on \( \mathbb{R}^n \), with unit ball, \( B \), say, with the property that if \( H \) and \( K \) are \((n - 1)\)-dimensional subspaces of \( \mathbb{R}^n \), \(|H \cap B| \leq c|K \cap B|\). We can take \( c \) to be \( \sqrt{6} \).

Let \( H \) be an \((n - 1)\)-dimensional subspace of \( \mathbb{R}^n \) and \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) be the unit vector normal to \( H \). Assume \( n \geq 2 \) and \(|a_i| < 1\), for all \( i \). Define \( f = f_H : \mathbb{R} \to [0, \infty) \) by \( f(r) = |(H + ra) \cap Q_n| \), so \( f(r) \) is the volume of the intersection of \( Q_n \) with a hyperplane parallel to \( H \), at distance \(|r|\) from \( H \). The assumption that \( n \geq 2 \), \(|a_i| < 1\) for all \( i \) ensures that \( H \) is not parallel to a face of \( Q_n \) and hence that \( f \) is continuous.

As in [3] we shall use probabilistic methods to obtain an expression for \(|H \cap Q_n| = f(0)\). Let \( X_1, \ldots, X_n \) be independent random variables, each uniformly distributed on \([-\frac{1}{2}, \frac{1}{2}]\), with respect to a probability, \( P \). Then the random vector \((X_1, \ldots, X_n)\) induces the usual measure on \( Q_n \), and, since \( f \) is continuous,

\[
\begin{align*}
f(r) &= \lim_{s \to 0} \frac{1}{2s} \int_{r-s}^{r+s} f(t) \, dt \\
&= \lim_{s \to 0} \frac{1}{2s} \left( x \in Q_n : \left| \sum_{i=1}^{n} a_i x_i - r \right| \leq s \right) \\
&= \lim_{s \to 0} \frac{1}{2s} P \left( \left| \sum_{i=1}^{n} a_i X_i - r \right| \leq s \right).
\end{align*}
\]
so \( f \) is the continuous probability density function of the random variable \( X = \sum a_i X_i \).

It was observed by Hensley that \( f \) attains its maximum at 0. This is also an immediate consequence of the Brünn-Minkowski inequality, namely,

\[
\left| \frac{1}{k} A + \frac{1}{k} B \right| \geq \left( \frac{1}{k} |A|^{1/k} + \frac{1}{k} |B|^{1/k} \right)^k
\]

for convex sets \( A \) and \( B \) in \( \mathbb{R}^k \). For, taking \( k = n - 1 \),

\[
A = H \cap (Q_n + ra) \subset H = \mathbb{R}^k, \quad \text{and} \quad B = H \cap (Q_n - ra) = -A,
\]

the convexity of \( Q_n \) gives

\[
H \cap Q_n \nrightarrow \frac{1}{2} (A + B) = \frac{1}{2} (A - A);
\]

so using the Brünn-Minkowski inequality gives

\[
f(0) = |H \cap Q_n| \geq \left| \frac{1}{2} (A - A) \right| \geq |A| = f(r).
\]

The upper bound, \( \sqrt{2} \), will thus apply to all sections of \( Q_n \) by hyperplanes, not merely those by subspaces.

Our principal aim is to prove best possible bounds on \( \|f\|_\infty = f(0) = |H \cap Q_n| \), with \((a_i)_{i=1}^n, (X_i)_{i=1}^n \) and \( f \) as above.

2. The lower bound. The lower bound will be an immediate consequence of the following simple probabilistic inequality, which compares a random variable with a uniformly distributed random variable of the same \( L_p \)-norm.

**Lemma 1.** Let \( Y \) be a random variable with a probability density function \( g: \mathbb{R} \rightarrow [0, \infty) \). Then for \( p > 0 \),

\[
\|g\|_\infty \|Y\|_p \geq \frac{1}{2} (p + 1)^{-1/p}.
\]

Equality holds in the above if and only if \( Y \) is uniformly distributed on some subinterval \([-t, t]\) of \( \mathbb{R} \).

**Proof.** Without loss of generality, we may assume that \( X \) is symmetric. Then setting

\[
G(x) = \int_0^x g(t) \, dt \quad \text{for } x \geq 0,
\]

we have \( G(0) = 0, G(\infty) = 1/2, \) and \( G(x) \leq x\|g\|_\infty \), so

\[
2^{-p} = 2(G(\infty))^{p+1} = 2 \int_0^\infty (G(x))^{p+1} \, dx
\]

\[
= 2(p+1) \int_0^\infty g(x) G(x)^p \, dx
\]

\[
\leq (p + 1) \|g\|_\infty^p \cdot 2 \int_0^\infty g(x) \cdot x^p \, dx
\]

\[
= (p + 1) \|g\|_\infty^p \|Y\|_p^p.
\]

There is equality if and only if \( g(t) = \|g\|_\infty \) or 0 almost everywhere.
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THEOREM 2. Every section of the unit cube, Q_n, in R" by an (n — 1)-dimensional subspace, H, has volume at least 1. The volume is 1 if and only if H is parallel to a face of the cube.

Proof. Plainly it suffices to show that, with (a_i)_i^n, (X_i)_i^n as above (|a_i| < 1 for all i) and f the continuous probability density function of X = Σa_iX_i, we have |H ∩ Q_n| = f(0) > 1. Since the X_i's are independent and identically distributed,

\[ EX^2 = \left( \sum a_i^2 \right) EX_i^2 = EX_i^2 = 1/12. \]

So \[ \|X\|_2 = 1/2\sqrt{3}, \] and the result follows by applying Lemma 1 with p = 2 and g = f.

We should remark that Theorem 2 was conjectured by A. Good, as was the following extension to subspaces of arbitrary dimension. Let U be a subspace of R"; then |U ∩ Q_n| ≥ 1. This extension was proved by Vaaler [4].

3. The upper bound. The proof of the principal result depends on the following lemma, whose proof we postpone until after the theorem itself.

Lemma 3.

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\sin t}{t} \right|^p dt \leq \frac{\sqrt{\pi}}{\sqrt{p}} \text{ if } p \geq 2, \]

and there is equality if and only if p = 2.

Theorem 4. Every section of the unit cube, Q_n, by an (n — 1)-dimensional subspace, H, has volume at most \( \sqrt{2} \). This upper bound is attained if and only if H contains an (n — 2)-dimensional face of Q_n.

Proof. Let (a_i)_i^n be as above and assume a_i ≥ 0 for all i. Suppose first that for some i, a_i > 1/\sqrt{2}; say a_1 > 1/\sqrt{2}. Consider H ∩ C_n, where C_n is the "cylinder"

\[ \{ x \in R^n : |x_i| \leq 1/2, 2 \leq i \leq n \}. \]

Let T: R^n → R^n be the orthogonal projection onto the subspace \( \{ x \in R^n : x_1 = 0 \} \), and S the orthogonal projection onto H. Then T(H ∩ C_n) is the unit cube in R^n-1 and

\[ |H ∩ C_n| = |T(H ∩ C_n)| \left\| Se \right\| \left\| TSe \right\| \left\| TSe \right\|, \]

where e is the unit vector (1,0,...,0) and \( \| \cdot \| \) is the usual Euclidean norm in R^n.

Since Se = (1 — a_1^2,...,—a_1a_n,...,—a_1a_n) and TSe = (0,—a_1a_2,...,—a_1a_n), we obtain

\[ |H ∩ C_n| = \frac{\left( 1 — a_1^2 \right)^2 + a_1^2 \sum a_i^2 a_i^2 \right)^{1/2}}{a_1^2 \sum a_i^2 a_i^2} = \frac{1}{a_1} < \sqrt{2}. \]

Hence |H ∩ Q_n| ≤ |H ∩ C_n| < \sqrt{2}.

Assume otherwise that a_i ≤ 1/\sqrt{2} for all i. Since, if some a_i is zero, the problem reduces to that in R^n-1, we may assume inductively that a_i > 0 for all i, and with this assumption the case of equality will follow if we show that \|H ∩ Q_n\| < \sqrt{2} unless n = 2 and a_1 = a_2 = 1/\sqrt{2}. With (X_i)_i^n independent random variables uniformly distributed on [—1/2, 1/2] and f the continuous p.d.f. of X = Σa_iX_i as
above, the random variable $a_iX_i$ has characteristic function
\[ \phi_i(t) = \left(2 \sin \frac{1}{2}a_it\right)/a_it, \]
and so the characteristic function of $X$ is
\[ \phi(t) = \prod_{i=1}^{n} \phi_i(t) = \prod_{i=1}^{n} \frac{2 \sin \frac{1}{2}a_it}{a_it} = \int_{-\infty}^{\infty} \cos xt \cdot f(x) \, dx. \]
By the standard Fourier inversion formula,
\[ |H \cap Q_n| = f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{2 \sin \frac{1}{2}a_it}{a_it} \, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{\sin a_it}{a_it} \, dt \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left| \frac{\sin a_it}{a_it} \right| \, dt. \]
Set $p_i = a_i^{-2}$ so that $p_i \geq 2$ for all $i$ and $\Sigma p_i^{-1} = 1$. Then by Hölder’s inequality with $n$ exponents,
\[ |H \cap Q_n| \leq \prod_{i=1}^{n} \left( \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\sin a_it}{a_it} \right| \, dt \right)^{1/p_i} = \prod_{i=1}^{n} \left( \frac{1}{a_i^{1/p_i}} \int_{-\infty}^{\infty} \left| \frac{\sin t}{t} \right| \, dt \right)^{1/p_i}. \]
By Lemma 3 this gives
\[ |H \cap Q_n| \leq \prod_{i=1}^{n} \left( \frac{\sqrt{2}}{a_i^{1/p_i}} \right)^{1/p_i} = \prod_{i=1}^{n} (\sqrt{2})^{1/p_i} = \sqrt{2}, \]
and there is equality only if $p_i = 2$ for all $i$, i.e., $n = 2$ and $a_1 = a_2 = 1/\sqrt{2}$.

**Proof of Lemma 3.** We divide the proof into two cases, $p \geq 4$ and $2 \leq p < 4$. (a) $p \geq 4$. Observe that
\[ \frac{\sin t}{t} = 1 - \frac{t^2}{6} + \frac{t^4}{120} - \frac{t^6}{7!} + \frac{t^8}{9!} - \cdots < 1 - \frac{t^2}{6} + \frac{t^4}{120} \quad \text{if } t^2 < 72, \]
\[ e^{-t^2/6} = 1 - \frac{t^2}{6} + \frac{t^4}{72} - \frac{t^6}{1296} + \frac{t^8}{6^4 \cdot 4!} - \frac{t^{10}}{6^5 \cdot 5!} + \cdots > 1 - \frac{t^2}{6} + \frac{t^4}{72} - \frac{t^6}{1296} \quad \text{if } t^2 < 30. \]
Since
\[ \frac{t^4}{72} - \frac{t^6}{1296} = \frac{t^4}{180} - \frac{t^6}{1296} \geq 0 \quad \text{if } t^2 \leq \frac{36}{5} \]
and $36/5 \leq \pi^2$, we have
\[ 0 \leq (\sin t)/t \leq e^{-t^2/6} \quad \text{if } t^2 \leq 36/5. \]
Let \( m = 6 / \sqrt{5} \). Then

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\sin t}{t} \right|^p dt < \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-pt^2/6} dt + \frac{2}{\pi} \int_{m}^{\infty} t^{-p} dt
\]

\[
= \frac{\sqrt{6}}{\sqrt{p} \cdot \sqrt{\pi}} + \frac{2}{\pi (p - 1) m^{p-1}}
\]

\[
\leq \frac{1}{\sqrt{p}} \left( \frac{\sqrt{6}}{\sqrt{\pi}} + \frac{4}{3\pi m^3} \right) \quad \text{for } p \geq 4
\]

\[
< \frac{\sqrt{2}}{\sqrt{p}}.
\]

(b) \( 2 < p < 4 \). Setting \( s = p/2 - 1 \), we wish to show that

\[
(1) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{\sin^2 t}{t^2} \right)^{1+s} dt \leq \frac{1}{\sqrt{1+s}} \quad \text{for } 0 \leq s < 1,
\]

with equality if and only if \( s = 0 \).

For \( n = 0, 1, 2, \ldots \) define

\[
\alpha_n = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-t^2/\pi} (1 - e^{-t^2/\pi})^n, \quad \beta_n = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} \left( 1 - \frac{\sin^2 t}{t^2} \right)^n dt.
\]

It is easily checked that

\[
\beta_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} dt = 1.
\]

Now

\[
\left( \frac{\sin^2 t}{t^2} \right)^{1+s} = \frac{\sin^2 t}{t^2} \left( 1 - \left( 1 - \frac{\sin^2 t}{t^2} \right) \right)^s
\]

\[
= \frac{\sin^2 t}{t^2} \sum_{0}^{\infty} \frac{(-1)^n s(s-1) \cdots (s-n+1)}{n!} \left( 1 - \frac{\sin^2 t}{t^2} \right)^n
\]

\[
= \frac{\sin^2 t}{t^2} \left( 1 - \sum_{1}^{\infty} \frac{|s(s-1) \cdots (s-n+1)|}{n!} \left( 1 - \frac{\sin^2 t}{t^2} \right)^n \right),
\]

since \( (-1)^n s(s-1) \cdots (s-n+1) \leq 0 \) if \( n \geq 1 \).

Hence by the monotone convergence theorem,

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{\sin^2 t}{t^2} \right)^{1+s} dt = \beta_0 - \sum_{1}^{\infty} \frac{|s(s-1) \cdots (s-n+1)|}{n!} \beta_n
\]

\[
= 1 - \sum_{1}^{\infty} \frac{|s(s-1) \cdots (s-n+1)|}{n!} \beta_n.
\]

Now we may write the right side of (1) as

\[
\frac{1}{\sqrt{1+s}} = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( e^{-t^2/\pi} \right)^{1+s} dt,
\]
and analogously to the above argument, we have

\[ \frac{1}{\sqrt{1 + s}} = 1 - \sum_{n=1}^{\infty} \frac{s(s - 1) \cdots (s - n + 1)}{n!} \alpha_n. \]

Hence it will suffice to show that \( \alpha_n < \beta_n \) for \( n \in \mathbb{N} \).

The first six values of \( \alpha_n \) and \( \beta_n \) (calculated using the residue theorem in the case of \( \beta_n \)), correct to two decimal places, are shown in Table 1.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha_n )</th>
<th>( \beta_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.29</td>
<td>0.33</td>
</tr>
<tr>
<td>2</td>
<td>0.16</td>
<td>0.22</td>
</tr>
<tr>
<td>3</td>
<td>0.11</td>
<td>0.17</td>
</tr>
<tr>
<td>4</td>
<td>0.08</td>
<td>0.15</td>
</tr>
<tr>
<td>5</td>
<td>0.07</td>
<td>0.13</td>
</tr>
<tr>
<td>6</td>
<td>0.05</td>
<td>0.12</td>
</tr>
</tbody>
</table>

It is easily checked that for \( n \geq 7 \),

\[ \frac{1}{\sqrt{2\pi} \cdot n} < \frac{1}{2\sqrt{\pi} \cdot \sqrt{n + 1}} - \frac{1}{2en}, \]

so to complete the proof it suffices to show that

(2) \( \alpha_n \leq \frac{1}{\sqrt{2\pi} \cdot n} \),

(3) \( \beta_n \geq \frac{1}{2\sqrt{\pi} \cdot \sqrt{n + 1}} - \frac{1}{2en} \) for \( n \geq 7 \).

We obtain these estimates as follows.

For \( 0 < x < 1 \),

\[ |\log x| \geq 1 - x + \frac{1}{2}(1 - x)^2 + \frac{1}{3}(1 - x)^3 + \frac{1}{4}(1 - x)^4 \]
\[ \geq 2(1 - x)^2 + (1 - x) - \frac{1}{2}(1 - x)^2 + \frac{1}{4}(1 - x)^4 \]
\[ = 2(1 - x)^2 + \frac{1}{2}x^2(1 - x)(3 - x) \]
\[ \geq 2(1 - x)^2. \]

Hence

\[ \alpha_n = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-x^2/\pi}(1 - e^{-x^2/\pi})^n \, dx \]
\[ = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2}(1 - e^{-u^2})^n \, du \]
\[ = \frac{1}{\sqrt{\pi}} \int_0^1 \frac{(1 - x)^n}{\sqrt{|\log x|}} \, dx \leq \frac{1}{\sqrt{2\pi}} \int_0^1 (1 - x)^{n-1} \, dx \]
\[ = \frac{1}{\sqrt{2\pi} \cdot n}, \text{ proving (2)}. \]
Now plainly,

\[ \beta_n = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} \left( 1 - \frac{\sin^2 t}{t^2} \right)^n \, dt \]

\[ \geq \frac{2}{\pi} \int_{1}^{\infty} \frac{\sin^2 t}{t^2} \left( 1 - \frac{\sin^2 t}{t^2} \right)^n \, dt \]

\[ \geq \frac{2}{\pi} \int_{1}^{\infty} \frac{\sin^2 t}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \, dt. \]

Observe that \((1/t^2)(1 - 1/t^2)^n\) is increasing for \(1 \leq t \leq \sqrt{n + 1}\) and decreasing for \(t \geq \sqrt{n + 1}\).

Hence, if \(n \geq 6\) so that \(\sqrt{n + 1} > 1 + \pi/2\),

\[ \int_{1 + \pi/2}^{\sqrt{n + 1}} \frac{\sin^2 t}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \, dt \geq \int_{1}^{\sqrt{n + 1} - \pi/2} \frac{\cos^2 t}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \, dt \]

and

\[ \int_{\sqrt{n + 1}}^{\infty} \frac{\sin^2 t}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \, dt \geq \int_{\sqrt{n + 1} + \pi/2}^{\infty} \frac{\cos^2 t}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \, dt. \]

So

\[ \int_{1}^{\infty} \frac{\sin^2 t}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \, dt \]

\[ \geq \frac{1}{2} \int_{1}^{\infty} \frac{1}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \, dt - \frac{1}{2} \int_{\sqrt{n + 1} - \pi/2}^{\sqrt{n + 1} + \pi/2} \frac{\cos^2 t}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \, dt \]

\[ \geq \frac{1}{2} \int_{1}^{\infty} \frac{1}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \, dt - \frac{\pi}{4} \max \left( \frac{1}{t^2} \left( 1 - \frac{1}{t^2} \right)^n \right) \]

\[ = \frac{1}{2} \int_{0}^{1} (1 - u^2)^n \, du - \frac{\pi}{4(n + 1)} \left( 1 - \frac{1}{n + 1} \right)^n \]

\[ = \frac{4^n}{2(2n + 1)} \left( \frac{2n}{n} \right)^{-1} - \frac{\pi}{4(n + 1)} \left( 1 - \frac{1}{n + 1} \right)^n \]

\[ \geq \frac{\sqrt{n}}{4\sqrt{n + 1}} - \frac{\pi}{4en}. \]

Hence

\[ \beta_n \geq \frac{1}{2\sqrt{\pi} \cdot \sqrt{n + 1}} - \frac{1}{2en}, \]

which is (3). \( \Box \)

The series comparison technique, used in part (b), above, was employed by Haagerup in his determination of the best constants in Khintchine's inequality [2]. We are indebted to J. Lindenstrauss for pointing out the similarity between the \(L_p\)-integral to be evaluated here and those of Haagerup.
Given the identity
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{\sin^2 t}{t^2} \right)^{1+s} dt = 1 + \sum_{n=1}^{\infty} (-1)^n s(s-1) \cdots (s-n+1) \frac{\beta_n}{n!} \beta_n, \quad 0 \leq s < 1,
\]
one could proceed with the proof of Lemma 3, part (b), in a number of ways. One possibility would be to consider the derivative with respect to \( s \), as follows.

Let
\[
I(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{\sin^2 t}{t^2} \right)^{1+s} dt, \quad J(s) = \frac{1}{\sqrt{1+s}}.
\]

Then
\[
I'(s) = \frac{d}{ds} \left( 1 + \sum_{n=1}^{\infty} (-1)^n s(s-1) \cdots (s-n+1) \frac{\beta_n}{n!} \beta_n \right)
\]
\[
= -\sum_{n=1}^{\infty} \frac{\beta_n}{n} + \sum_{n=2}^{\infty} s \cdot \frac{(1-s) \cdots (n-1-s)}{n!} \beta_n \cdot \sum_{k=1}^{n-1} \frac{1}{k-s}
\]
\[
= -\sum_{n=1}^{\infty} \frac{\beta_n}{n} + \sum_{n=2}^{\infty} s \cdot \frac{1}{n} \prod_{j=1}^{n-1} \left( 1 - \frac{s}{j} \right) \beta_n \sum_{k=1}^{n-1} \frac{1}{k-s}
\]
\[
\leq -\sum_{n=1}^{\infty} \frac{\beta_n}{n} + \sum_{n=2}^{\infty} s \beta_n \cdot (3 + \log(n-1)) \quad \text{for } 0 \leq s \leq \frac{1}{2}.
\]

A simple computation shows that \( \beta_n \leq 1/\sqrt{n} \) for all \( n \), and that the above then gives
\[
I'(s) \leq -\sum_{n=0}^{\infty} \frac{\beta_n}{n} + 10s \quad \text{for } 0 \leq s \leq \frac{1}{2}.
\]

Substituting the values of \( \beta_n \) for \( 1 \leq n \leq 6 \) gives
\[
-\sum_{n=0}^{\infty} \frac{\beta_n}{n} < -0.58,
\]
so
\[
I'(s) < -1/2 \leq J'(s) \quad \text{for } 0 \leq s \leq 0.008.
\]

This gives \( I(s) < J(s) \) for \( 0 < s \leq 0.008 \). The fact that \( I \) and \( J \) are decreasing functions enables us to prove that \( I(s) < J(s) \) for \( 0.008 < s \leq 1 \), using finitely many iterations of the following scheme. Let \( s_0 = 0.008 \). Using sufficiently accurate values for \( \beta_n \) for sufficiently many \( n \), find \( t_0 \) so that \( I(s_0) \leq t_0 < J(s_0) \). Choose \( s_1 \) so that \( J(s_1) \geq t_0 \) and repeat to obtain \( t_1, s_2, t_2, s_3, \ldots \). For some \( k \) we obtain \( s_k > 1 \), and the result is proven.

4. Additional remarks. Despite the similarity of Theorem 4 to the classical Khintchine inequality, the two results do not seem to be readily deducible from one another. However, Theorem 4 and Lemma 1 together provide a simple analogue of Khintchine's inequality for \( 0 < p < 2 \) for independent, uniformly distributed random variables.
Proposition 5. Let $X_1, \ldots, X_n$ be independent random variables, uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$ and $p > 0$. Then
\[
\left\| \sum a_i X_i \right\|_p \geq \frac{\sqrt{3}}{\sqrt{2} \left( p + 1 \right)^{1/p}} \left\| \sum a_i X_i \right\|_2 = \frac{1}{2 \sqrt{2} \left( p + 1 \right)^{1/p}} \left( \sum a_i^2 \right)^{1/2}
\]
for any sequence, $(a_i)_i$, of scalars.

It is natural to ask what is the maximum volume (in terms of $k$ and $n$) of the intersection of $Q_n$ with a $k$-dimensional subspace of $\mathbb{R}^n$. While for $k = 1$ the answer is trivially $\sqrt{n}$ and for $k = n - 1$ the answer is provided by Theorem 3, it is not clear what should be the result for general $k$.

This work will form part of a doctoral thesis being written under the supervision of Dr. B. Bollobás, whose constant advice and encouragement have been invaluable.

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