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KINEMATIC FORMULAS FOR WEYL'S CURVATURE INVARIANTS OF SUBMANIFOLDS IN COMPLEX PROJECTIVE SPACE

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ABSTRACT. It is shown in [**5**] that Weyl's curvature invariants $k_{2p}(M)$ can be expressed by $\gamma_q \wedge F^{m-q}[M]$, where M is a $2m$-dimensional Kähler submanifold with compact closure in a space of constant holomorphic curvature, γ_q is the qth Chern form of M and F is the Kähler form of M. In this paper, we shall show that each $\gamma_q \wedge F^{m-q}[M]$ is expressible in terms of F and $k_{2p}(M)$. Using this result, we get kinematic formulas for $k_{2p}(M)$ from Shifrin's [**8**] kinematic formulas for Chern classes.

1. Introduction. In this paper we rely on the notation, terminology and results in [**3** and **5**]. Let M be a Kähler submanifold of complex dimension m with compact closure in a space $N(\lambda)$ of constant holomorphic curvature 4λ ($\lambda > 0$). Let F be the Kähler form of M and put $K = R^M - R^{N(\lambda)}$ on M, where R^M and $R^{N(\lambda)}$ are the curvature tensors of M and $N(\lambda)$, respectively. Then K is a Kähler curvature-like tensor field on M. For $p = 1, \ldots, m$, let $k_{2p}(M)$ be the curvature invariant defined for the tensor field K by Weyl in [**9**] (see also (7.6) in [**4**]). For $p = 1, \ldots, m$, let γ_p be the pth Chern form of M. Put $\Gamma_p(M) = (\gamma_p \wedge F^{m-p})[M]$ for $1 \le p \le m$ and $\Gamma_0(M) = F^m[M]$. Then we have (see (3.26) and (3.27) in [**5**] and also Lemma 4.1 in [**6**]).

LEMMA 1 (A. GRAY [**5**]). *For $1 \le p \le m$, we have*

$$k_{2p}(M) = \sum_{q=0}^{p} \frac{(2\pi)^p}{(m-p)!} \binom{m-q+1}{p-q} \left(-\frac{\lambda}{\pi}\right)^{p-q} \Gamma_q(M). \tag{1.1}$$

Conversely we will get

LEMMA 2. *For $1 \le p \le m$, we have*

$$\Gamma_p(M) = \sum_{q=1}^{p} \frac{(m-q)!}{(2\pi)^q} \binom{m-q+1}{p-q} \left(\frac{\lambda}{\pi}\right)^{p-q} k_{2q}(M) + \binom{m+1}{p} \left(\frac{\lambda}{\pi}\right)^{p} F^m[M]. \tag{1.2}$$

Next we state Shifrin's kinematic formulas (see (3.5) and (4.4) in [**8**]).

THEOREM 1 (T. SHIFRIN [**8**]). *Let M^m and N^l be Kähler submanifolds of complex dimensions m and l, respectively, in a Fubini-Study space $N(\lambda)$. Assume*

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that $k = m + l - n \geq 1$. *Then, for* $1 \leq h \leq k$,

(1.3)

$$\int_{U(n+1)\to g} \Gamma_h(M^m \cap gN^l)\, dg$$
$$= \mathrm{Vol.}(U(n+1)) \sum_{0\leq i, 0\leq j}^{i+j\leq n} (-1)^h \binom{n+h-i-j}{n} \left(\frac{\lambda}{\pi}\right)^{n+h-i-j} \Gamma_i(M^m)\Gamma_j(N^l)$$

whenever both sides make sense. Let $G(l,n)$ *be the space of projective* l*-planes in* CP^n $(= N(\lambda))$. *Then, for* $0 \leq h \leq k$,

$$(1.4)\quad \int_{G(l,n)\ni L} \Gamma_h(M^m \cap L)\, dL$$
$$= \mathrm{Vol.}(G(l,n)) \sum_{j=0}^{h} (-1)^{h-j} \binom{n+h-l-j-1}{h-j} \left(\frac{\lambda}{\pi}\right)^{n+h-l-j} \Gamma_j(M^m),$$

whenever both sides makes sense.

The normalized Kähler form $\omega = \lambda F/\pi$ is used in [**8**] so that the coefficients in Shifrin's original kinematic formulas are slightly different from those in the above formulas. Using Lemmas 1 and 2, we obtain from Theorem 1 the following kinematic formulas for Weyl's curvature invariants k_{2p}.

THEOREM 2. *Let* M^m *and* N^l *be Kähler submanifolds with compact closure in a Fubini-Study space* $N(\lambda)$ *of constant holomorphic curvature* 4λ $(\lambda > 0)$. *Assume that* $k = m + l - n \geq 1$. *Then, for* $1 \leq p \leq k$,

$$(1.5)\quad \int_{U(n+1)} k_{2p}(M^m \cap gN^l)\, dg = \mathrm{Vol.}(U(n+1)) \sum_{0\leq a,b}^{a+b\leq p} F_{pab} k_{2a}(M^m) k_{2b}(N^l),$$

$$(1.6)\quad \int_{G(l,n)} k_{2p}(M^m \cap L)\, dL = \mathrm{Vol.}(G(l,n)) \left(\frac{\lambda}{\pi}\right)^{n-l} \frac{(m-p)!}{(k-p)!} k_{2p}(M^m),$$

where

(1.7)

$$\begin{cases} f_{pab} = (-1)^p \left(\dfrac{\lambda}{\pi}\right)^n \dfrac{(2\sqrt{\lambda})^{2(p-a-b)}}{(p-a-b)!} \dfrac{(m-a)!(l-b)!}{(k-p)!} \dfrac{(n+k-a-b+2)!}{(n+k-p+2)!}, \\ F_{pab} = f_{pab} \quad \textit{for } a \geq 1 \textit{ and } b \geq 1, \\ F_{p0b} = f_{p0b}/m! \quad \textit{for } p \geq b \geq 1, \\ F_{pa0} = f_{pa0}/l! \quad \textit{for } p \geq a \geq 1, \qquad F_{p00} = f_{p00}/m!l!. \end{cases}$$

Weyl's curvature invariants have relations with other invariants. In fact, let $\mu_{2p}(K)$ be the Chern's curvature invariants for the curvature-like tensor K (see [**2**]). Put $(2p-1)!! = (2p-1)\cdot(2p-3)\cdots 3\cdot 1$. Then it holds that

$$\mu_{2p}(K) = \left(\binom{2m}{2p}(2p-1)!!\right)^{-1} k_{2p}(M).$$

Also, $k_{2p}(M)$ are closely related to mean curvatures of M in $N(\lambda)$ (see, for example, [**1**]).

2. Proof of Lemma 2. We may put $k_0(M) = (1/m!)\Gamma_0(M)$. Let, for $0 \le p \le m$ and $0 \le q \le m$,

$$a_p = \frac{(m-p)!}{(2\pi)^p}\left(-\frac{\pi}{\lambda}\right)^p k_{2p}(M), \qquad b_q = \left(-\frac{\pi}{\lambda}\right)^q \Gamma_q(M).$$

Then, from (1.1) we have, for $0 \le p \le m$, $a_p = \sum_{q=0}^{p} \binom{m+1-q}{p-q} b_q$. Now, we use formula (7a) (on p. 53 in [**7**]), and get

$$b_p = \sum_{q=0}^{p} (-1)^{p+q} \binom{m+1-q}{p-q} a_q.$$

This implies (1.2).

3. Proof of Theorem 2. First we will show (1.5). If we write (1.3) as

$$\int \Gamma_h(M^m \cap gN^l)\, dg = \sum_{0\le i,\ 0\le j}^{i+j\le h} D_{hij}\Gamma_i(M^m)\Gamma_j(N^l),$$

by means of (2.1) and (2.2), we have, for $1 \le p \le k$,

$$\int k_{2p}(M^m \cap gN^l)\, dg = \sum_{h=0}^{p} \sum_{0\le i,\ 0\le j}^{i+j\le h} \sum_{a=0}^{i}\sum_{b=0}^{j} A_{ph} D_{hij} B_{ia} B_{jb} k_{2a}(M^m) k_{2b}(N^l).$$

Thus we obtain

$$\text{Vol.}(U(n+1))F_{pab} = \sum_{h=a+b}^{p} \sum_{a\le i,\ b\le j}^{i+j\le h} A_{ph} D_{hij} B_{ia} B_{jb}. \tag{3.1}$$

Put

$$f_{pab} = (-1)^p \left(\frac{\lambda}{\pi}\right)^n (2\lambda)^{p-a-b} \frac{(m-a)!(l-b)!}{(k-p)!} I_p, \tag{3.2}$$

(3.3)

$$I_p = \sum_{h=a+b}^{p} \sum_{a\le i,\ b\le j}^{i+j\le j} \binom{k-h+1}{p-h} \binom{n+h-i-j}{n} \binom{m-a+1}{i-a} \binom{l-b+1}{j-b}.$$

Using (1.1)–(1.3), from (3.1) we get, for $1 \le a,\ 1 \le b,\ a+b \le p$,

$$\begin{aligned} F_{pab} &= f_{pab}, & F_{p0b} &= f_{j0b}/m!, \\ F_{pa0} &= f_{pa0}/l!, & F_{p00} &= f_{p00}/m!l!. \end{aligned} \tag{3.4}$$

To calculate I_p, we write them as

$$I_p = \sum_{h'=0}^{p'} \sum_{t=0}^{h'} \binom{k'-h+1}{p'-h} \binom{n+h-t}{n} \binom{m'+l'+2}{t}, \tag{3.5}$$

where $m' = m-a$, $l' = l-b$, $k' = k-a-b$ and $p' = p-a-b$. We use the identity

$$\sum_{i+j=t} \binom{m'+1}{i} \binom{l'+1}{j} = \binom{m'+l'+2}{t}$$

to get (3.5). We know the identities (see, for example, [**7**])

$$\sum_{h=t}^{p'} \binom{k'-h+1}{p'-h} \binom{n+h-t}{n} = \binom{n+k'-t+2}{p'-t}, \tag{3.6}$$

$$\sum_{t=0}^{p'} \binom{n+k'+2}{t} \binom{n+k'+2-t}{p'-t} = 2^{p'} \binom{n+k'+2}{p'}. \tag{3.7}$$

Hence, from (3.5), it follows that

$$I_p = 2^{p'} \binom{n+k'+2}{p'} = 2^{p-a-b} \binom{n+k-a-b+2}{p-a-b}.$$

Combining this with (3.2) and (3.4), we get (1.7).

Next, we prove (1.6). We may put, for $1 \leq p \leq k$,

$$\int \Gamma_p(M^m \cap L)\, dL = \sum_{q=0}^{p} E_{pq} \Gamma_q(M^m),$$

$$\int k_{2p}(M^m \cap L)\, dL = \sum_{q=1}^{p} G_{pq} k_{2q}(M^m).$$

Using (2.1) and (2.2), we get

$$G_{pq} = \sum_{h=q}^{p} \sum_{i=q}^{h} A_{ph} E_{hi} B_{iq}. \tag{3.8}$$

Assume that $1 \leq q < p$. Then, using (1.1), (1.2) and (1.4), we get

$$G_{pq} = \mathrm{Vol.}(G(l,n))(2\lambda)^{p-q} \left(\frac{\lambda}{\pi}\right)^{n-l} \frac{(m-q)!}{(k-p)!} H_{pq}, \tag{3.9}$$

(3.10)

$$H_{pq} = \sum_{i'=0}^{p'} \sum_{h'=i'}^{p'} (-1)^{p'-i'} \binom{k'-h'+1}{p'-h'} \binom{n-l+h'-i'-1}{h'-i'} \binom{m'+1}{i'},$$

where $i' = i-q$, $p' = p-q$, $h' = h-q$ and $m' = m-q$. Similar to (3.6), we have

$$\sum_{h'=i'}^{p'} \binom{k'-h'+1}{p'-h'} \binom{n-l+h'-i'-1}{h'-i'} = \binom{m'-i'+1}{m'-p'+1}.$$

Hence, from (3.10), it follows that

$$H_{pq} = (-1)^{p'} \sum_{i'=0}^{p'} (-1)^{i'} \binom{m'+1-i'}{m'+1-p'} \binom{m'+1}{i'} = 0.$$

This gives $G_{pq} = 0$ for $1 \leq q < p$. Similarly we get $G_{p0} = 0$. Thus we obtain $G_{pq} = 0$ for $0 \leq q < p$. On the other hand, we have

$$G_{pp} = A_{pp} E_{pp} B_{pp} = \mathrm{Vol.}(G(l,n)) \left(\frac{\lambda}{\pi}\right)^{n-l} \frac{(m-p)!}{(k-p)!}.$$

Thus we proved (1.6).

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