NONARCHIMEDEAN $C^\#(X)$

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ABSTRACT. Let $E$ be a nonarchimedean rank-one valued field, and $X$ an ultraregular topological space. We consider the Gelfand subalgebra $C^\#(X, E)$ of the algebra of all $E$-valued continuous functions on $X$, and the algebra $F(X, E)$ consisting of those $E$-valued continuous functions $f$ for which there exists a compact set $K \subset X$ such that $f(X - K)$ is finite. We obtain some characterizations of $C^\#(X, E)$, analogous to those obtained in the real case, which we use to find conditions that imply the equality $C^\#(X, E) = F(X, E)$ holds.

For $T$ a completely regular topological space and $C(T, R)$ the space of continuous real valued functions on $T$, let $C^\#(T, R)$ denote the Gelfand subalgebra of $C(T, R)$, consisting of all $f \in C(T, R)$ with the property that, for every maximal ideal $m$ of $C(T, R)$, there exists an $r \in R$ such that $(f - r) \in m$. We shall denote by $F(T, R)$ the subalgebra of $C(T, R)$ consisting of those $f \in C(T, R)$ for which there exists a compact $K \subset T$ ($K$ depending on $f$) such that $f(T - K)$ is finite. The basic properties of $C^\#(T, R)$ are established in \[NR, C and SZ\], and they are summarized in \[H, Theorem 2.1\]. It follows from \[SZ and N\] that if $T$ is a real compact and locally compact space or if $T$ is a normal metacompact and locally compact space, then $C^\#(T, R) = F(T, R)$.

Now, let $E$ be a nonarchimedean rank-one valued field (which we do not assume to be complete), and $X$ an ultraregular topological space. Let $C(X, E), C^\#(X, E)$ and $F(X, E)$ stand for the nonarchimedean analogue of the concepts defined above. As the main result in \[D_2\] we saw, that if one assumes $X$ is paracompact and locally compact, then $C^\#(X, E) = F(X, E)$. In the present paper, by using a nonarchimedean analogue of \[H, Theorem 2.1\], we obtain all the results of \[D_2\], with a weaker hypothesis in the case of the main result, as relatively simple corollaries.

If $A$ is a unitary commutative ring, $M(A)$ will denote the set of all maximal ideals of $A$ endowed with the Zariski topology (or hull-kernel topology). Thus $M(C(X, E)) = \beta_0 X$ (the Banaschewski compactification of $X$). For the rest we shall use the notation of \[BB\] except that we shall use “cl” to denote topological closure.

Let $\overline{E}$ be the completion of the valued field $E$.

**Lemma 1.** $C^\#(X, E) = C^\#(X, \overline{E}) \cap C(X, E)$.
PROOF. It suffices to take into account that, by the ultraregular analogue of the Gelfand-Kolmogoroff theorem (see [BB, Theorem 6]), one has a bijection (in fact a homeomorphism) \( \mathcal{M}(C(X, E)) \rightarrow M(C(X, E)) \) sending each maximal ideal \( m^- \) of \( C(X, E^-) \) to \( m^- \cap C(X, E) \).

REMARK. In dealing with \( C^\#(X, E) \), sometimes this lemma can allow us to assume, without loss of generality, that \( E \) is a complete field. In particular, the referee had previously pointed out to us that something like this was necessary to close a gap in the second part of the proof of Proposition 1 of [D2], as the argument we use there does not make apparent that, for incomplete fields, \( f^\beta : M \rightarrow L \) does indeed carry all of \( M \) to \( L \). Now the lemma does close that gap.

Let \( N \) denote the discrete space of positive integers.

**Lemma 2.** The functions in \( C^\#(N, E) \) are exactly those with finite range.

**Proof.** If \( f \in C(N, E) \) has a finite range \( f(N) = \{\lambda_1, \ldots, \lambda_n\} \), then \( \Pi(f - \lambda_i) = 0 \). Hence for every maximal ideal \( m \) of \( C(N, E) \) one has \( \Pi(f - \lambda_i) \in m \) and so \( f - \lambda_i \in m \) for some \( \lambda_i \). Thus \( f \in C^\#(N, E) \). In order to see the converse, take \( f \in C(N, E) \) with \( f(N) \) infinite, and for any \( \lambda \in E \) set \( Z_\lambda = f^{-1}(E - \{\lambda\}) \).

The family \( (Z_\lambda) \) has the finite intersection property, so there is a maximal ideal \( m \) of \( C(N, E) \) such that \( Z_\lambda \cap Z_{\lambda'} = \emptyset \) for every \( \lambda, \lambda' \in E \) different. For any \( \lambda \in E \) one has \( Z(f - \lambda) \cap Z_\lambda = \emptyset \) and hence \( \{f - \lambda\} \notin m \), so \( f \notin C^\#(N, E) \).

**Lemma 3.** Let \( f \in C(X, E) \) and assume \( f(X) \) is not precompact. Then there is a clopen partition \((U_i)_{i \in I}\) of \( X \) and there exist \( x_i \in U_i \) such that \( f(\{x_i/i \in I\}) \) is infinite.

**Proof.** If \( f(X) \) is not precompact, then there exists \( \varepsilon > 0 \) such that \( f(X) \) has no finite covers by \( \varepsilon \)-radius spheres. For every \( \alpha \in f(X) \) set \( B(\alpha) = \{\mu \in E/|\mu - \alpha| \leq \varepsilon\} \). Since any two spheres \( B(\alpha) \) are either equal or disjoint, there exist \( (\alpha_i)_{i \in I} \), \( I \) being an infinite set, such that the sets \( B(\alpha_i) \) are pairwise disjoint and form a clopen cover of \( f(X) \). Now set \( U_i = f^{-1}(B(\alpha_i)) \) and choose \( x_i \in X \) such that \( f(x_i) = \alpha_i \).

It will be said that a subset \( S \) of \( X \) is \( C \)-embedded in \( X \) with respect to \( E \) if every continuous function from \( S \) into \( E \) has a continuous extension to \( X \).

In analogy with the Hewitt real compactification of a completely regular space, we let \( \nu_0 X \) be the set of all \( p \in \beta_0 X \) such that, for every sequence \((V_n)\) of neighbourhoods of \( p \) in \( \beta_0 X \), \( \bigcap V_n \cap X \neq \emptyset \).

Now we shall state the nonarchimedean analogue of [H, Theorem 2.1] (see also [SZ, C and NR]).

**Theorem 1.** If \( f \in C(X, E) \), then the following are equivalent:

(a) \( f \in C^\#(X, E) \),
(b) \( f(D) \) is finite for every copy \( D \) of \( N \) which is \( C \)-embedded in \( X \) with respect to \( E \),
(c) \( f(Z) \) is compact for every \( E \)-zero-set \( Z \) in \( X \),
(d) \( f(X) \) is compact and for every \( \lambda \in E \), \( \text{cl}_{\beta_0 X} Z(f - \lambda) = Z(\beta_0 f - \lambda) \),
(d') \( f(X) \) is relatively compact and for every \( \lambda \in E \), \( \text{cl}_{\beta_0 X} Z(f - \lambda) = Z(\beta_0 f - \lambda) \).
Moreover if $E$ has nonmeasurable cardinality, the above conditions are also equivalent to

(e) $f(X)$ is compact and, for every $p \in \beta_0 X - \nu_0 X$, there is a neighbourhood of $p$ in $\beta_0 X$ on which $\beta_0 f$ is constant.

PROOF. (a)$\Rightarrow$(b). The restriction map $C(X,E) \to C(D,E)$, $f \mapsto f|_D$ is a surjective $E$-algebra homomorphism, so if $f \in C^*(X,E)$ then $f|_D \in C^*(D,E)$, and hence by Lemma 2 $f(D)$ is finite.

(b)$\Rightarrow$(c). Let $Z = Z(g), g \in C(X,E)$. From Lemma 3, $f(X)$ is precompact and so $f(Z)$ is too. To show $f(Z)$ is compact, we are going to see that $f(Z) = \text{cl}_E f(Z)$.

Assume, otherwise, that there exists $\lambda \in \text{cl}_E f(Z) - f(Z)$. Then the set $A = \{x \in X/|g(x)| < |f(x) - \lambda|\}$ is a clopen set such that $Z \subset A$ and $Z(f - \lambda) \subset X - A$, and the continuous function $1/(f - \lambda): A \to E^\sim$ does not have precompact range. By Lemma 3, there is a clopen partition $(U_i)_{i \in I}$ of $A$ and there is $x_i \in U_i$ such that $1/(f - \lambda)$ takes infinite values on the set $\{x_i/i \in I\}$, so $f(\{x_i/i \in I\})$ is infinite. One immediately sees that this is contradictory to the assumption that (b) holds.

(c)$\Rightarrow$(d). Let $p \in Z(\beta_0 f - \lambda)$. By [BB, Theorem 6], $m_p = \{g \in C(X,E)/p \in \text{cl}_{\beta_0 X} Z(g)\}$ is a maximal ideal and $h \in C(X,E)$ belongs to $m_p$ iff $Z(h)$ meets $Z(g)$ for each $g$ in $m_p$. By (c), $f(Z(g)) = \beta_0 f(\text{cl}_{\beta_0 X} Z(g))$ for each $g$ in $m_p$. Thus there exists $x \in Z(g)$ such that $f(x) = \beta_0 f(p) = \lambda$. Hence $Z(f - \lambda) \cap Z(g) \neq \emptyset$ for any $g$ in $m_p$, from which it follows that $p \in \text{cl}_{\beta_0 X} Z(f - \lambda)$ by [BB, Theorem 6]. Thus $Z(\beta_0 f - \lambda) \subset \text{cl}_{\beta_0 X} Z(f - \lambda)$. The reverse inclusion is clear.

(d)$\Rightarrow$(d'). It is obvious.

(d')$\Rightarrow$(a). If $p \in \beta_0 X$, then $p \in Z(\beta_0 f - \beta_0 f(p)) = \text{cl}_{\beta_0 X} Z(f - \beta_0 f(p))$ and so $(f - \beta_0 f(p)) \in m_p$.

(b)$\Rightarrow$(e). By the above $f(X)$ is compact. Let $p \in \beta_0 X - \nu_0 X$. Then there is a sequence $(V_n)$ of clopen neighbourhoods of $p$ in $\beta_0 X$ such that $\bigcap V_n \cap X = \emptyset$ and such that $V_n \cap X \supset V_{n+1} \cap X$. Assume there is no neighbourhood of $p$ in $\beta_0 X$ on which $\beta_0 f$ is constant. Hence there is a sequence $(x_k), x_k \in (V_{n_k} - V_{n_k+1}) \cap X$ for some increasing sequence $(n_k)$, such that $f(x_k) \neq f(x_j)$ for $k \neq j$. The set $D = \{x_k/k \in N\}$ is a copy of $N$, $C$-embedded in $X$ with respect to $E$, and $f(D)$ is infinite. This is contradictory to the assumption that (b) holds. So there is a neighbourhood of $p$ in $\beta_0 X$ on which $\beta_0 f$ is constant.

(e)$\Rightarrow$(a). By Lemma 1 we may suppose that $E$ is a complete field, and we may also assume that $E$ is infinite, as $C^*(X,E) = C(X,E)$ if $E$ is finite. Now we make the additional assumption that $E$ has nonmeasurable cardinality. By [BB, Theorem 15] one has $\nu_0 X = \nu_E X$, that is, $\nu_0 X$ is the set of all maximal ideals of $C(X,E)$ of codimension one. To prove $f \in C^*(X,E)$ it suffices to see that if $p \in \beta_0 X - \nu_0 X$ and $\lambda = \beta_0 f(p)$, then $(f - \lambda) \in m_p$ or, equivalently, that $p \in \text{cl}_{\beta_0 X} Z(f - \lambda)$. This last condition is true as by hypothesis $\beta_0 f$ is constantly equal to $\lambda$ on a neighbourhood of $p$ in $\beta_0 X$.

The following result due to K. Nowinski [N, Theorem 2] will be used in the next corollary:

"If $f : Z \to Y$ is a closed continuous map from a metacompact locally compact Hausdorff space $Z$ to a compact space $Y$, then there exists a compact $K \subset Z$ such that $f(Z - K)$ is finite".
But first we need a lemma:

**Lemma 4.** Let $f \in C^\#(X, E)$ and assume $X$ is ultranormal. Then $f$ is a closed map.

**Proof.** Let $B$ be a closed subset of $X$, $p \in \text{cl}_{\beta_0}X$ and $\lambda = \beta_0f(p)$. Assume $Z(f - \lambda) \cap B = \emptyset$. Since $X$ is ultranormal, there is a clopen subset $A$ of $X$ such that $Z(f - \lambda) \subset A$ and $B \subset X - A$. Let $e_A$ be the $E$-valued characteristic function of $A$. Since $Z(e_A) \supset B$, one has $e_A \in m_p$. On the other hand, since $\lambda = \beta_0f(p)$ and $f \in C^\#(X, E)$, one also has $(f - \lambda) \in m_p$. As $Z(f - \lambda) \cap Z(e_A) = \emptyset$, we get a contradiction. So $f(B) = \beta_0f(\text{cl}_{\beta_0}X B)$ is a closed subset of $E$.

**Corollary 1.** In order that $C^\#(X, E) = F(X, E)$ it suffices that any of the following conditions holds:

(a) $X$ is an ultranormal metacompact and locally compact space,
(b) $E$ is a complete field with nonmeasurable cardinal and $X$ is an $E$-replete locally compact space,
(c) the valuation of $E$ is trivial.

**Proof.** From (a) and (b) of Theorem 1 (cf. [Di, Proposition 6]), it is evident that $F(X, E) \subset C^\#(X, E)$, so we shall prove the converse. Take $f \in C^\#(X, E)$. Assume that (a) holds. From Lemma 4 and Theorem 1, $f$ is a closed map and $f(X)$ is compact, so $f \in F(X, E)$ by the above result of Nowinski. If the valuation of $E$ is trivial then, for $f \in C^\#(X, E)$, $f(X)$ is compact and thus finite, and so $C^\#(X, E) \subset F(X, E)$. Thus it only remains to prove the inclusion in case $E$ is an infinite field and (b) holds. In this situation one has $X = \nu_E X = \nu_0 X$, and the result follows directly from the equivalence of conditions (a) and (e) in Theorem 1, taking into account the compactness of $\beta_0 X - X$.

**Remark.** Note that an ultraregular paracompact locally compact space satisfies condition (a) in Corollary 1 (see [E, §1 and V, p. 40]). So Corollary 1 strengthens [D2, Theorem].

**Examples.** Example 1 below shows that no condition (a), (b) or (c) in Corollary 1 is necessary in order to have $C^\#(X, E) = F(X, E)$. On the other hand, Examples 2 and 3 prove that neither metacompactness nor local compactness can be dropped in (a).

Let $Q_p$ be the field of the $p$-adic numbers and $\bar{Q}_p$ the algebraic closure of $Q_p$. Extend to $\bar{Q}_p$ the $p$-adic absolute value. Let $\Omega_p$ be the completion of $\bar{Q}_p$ and again extend to $\Omega_p$ the absolute value on $\bar{Q}_p$. In this way $\Omega_p$ is a complete field with respect to a (nonarchimedean) absolute value which extends the $p$-adic absolute value on $Q$. Moreover, $\Omega_p$ is algebraically closed and therefore it is not locally compact.

For the following examples set $E = \Omega_p$.

**Example 1.** (see [GJ, p. 123]). Let $W$ be the set of all ordinals less than the first uncountable ordinal endowed with the interval topology. $W$ is an ultranormal locally compact space which is neither metacompact nor $E$-replete. Nevertheless, $C^\#(W, E) = F(W, E) = C(W, E)$.

**Example 2.** Let $X = \Omega_p$. Then $X$ is an ultranormal metacompact $E$-replete space, but we shall see that $C^\#(X, E) \neq F(X, E)$. To see this, set $X_0 = \{\alpha \in X/1 \leq |\alpha|\}$ and $X_n = \{\alpha \in X/1/p^n \leq |\alpha| < 1/p^{n-1}\}$, $n = 1, 2, \ldots$. As the sets
$X_n (n = 0, 1, 2, \ldots)$ are clopen, the function $f: X \to E$, given by $f(0) = 0$ and $f(\alpha) = p^n$ for $\alpha \in X_n$, is continuous, and it is clear that $f \notin F(X,E)$. On the other hand, if $D$ is a $C$-embedded copy of $N$, then $D - \{0\}$ is bounded away from 0, so $f(D)$ is finite. From (a) and (b) of Theorem 1, it follows that $f \in C^\#(X,E)$.

**Example 3.** Let $X = W \times W$. Then $X$ is an ultranormal locally compact space, but $C^\#(X,E) = C(X,E) \neq F(X,E)$. (To see that $X$ is an ultranormal space, note that, by Glicksberg's theorem (or [GJ, 8 M2]), one has $\beta X = \beta W \times \beta W$. Hence $\beta X$ is an ultraregular compact space, and so the large inductive dimension of $\beta X$, Ind($\beta X$), is 0. On the other hand, since $X$ is a normal space, one has Ind($X$) = Ind($\beta X$) (see [I, Theorem 8, p. 100]). Thus Ind($X$) = 0, and so $X$ is an ultranormal space.)

**Example 4 (see [N, Example 3 and D3, Example 3]).** Let $D$ be a discrete space of power $c$, $D^*$ the one-point compactification $D \cup \{w_1\}$ of $D$, $N^*$ the one-point compactification $N \cup \{w\}$ of $N$, and $X = N^* \times D^* - \{(w, w_1)\}$. Then $X$ is an ultraregular metacompact space which is neither ultranormal nor $E$-replete, nevertheless $C^\#(X,E) = F(X,E)$.

As usual, $C_K(X,E)$ will denote the ideal of $C(X,E)$ consisting of those functions with compact support. An ideal $J$ of $C(X,E)$ will be called free if
\[
\bigcap \{Z(f) / f \in J\} = \emptyset.
\]

**Corollary 2.** Assume that any of the following conditions holds:
(a) $X$ is ultranormal metacompact and locally compact,
(b) $E$ is a complete field with nonmeasurable cardinal and $X$ is $E$-replete,
(c) the valuation on $E$ is trivial.

Then $C_K(X,E) = \bigcap \{m/m$ is a free maximal ideal of $C(X,E)\}$.

**Proof.** As in [GJ, 4D] $C_K(X,E)$ is contained in every free (maximal) ideal of $C(X,E)$. To prove the reverse inclusion take $f$ belonging to every free maximal ideal of $C(X,E)$. It is clear that $f \in C^\#(X,E)$. First assume (b). Then from Theorem 1, for any $p \in \beta_0 X - X$ there is a neighbourhood of $p$ in $\beta_0 X$ on which the function $\beta_0 f$ vanishes, so $\beta_0 f$ vanishes on an open neighbourhood of $\beta_0 X - X$, whence $f \in C_K(X,E)$. Now assume (a) or (c). Then from Corollary 1, $f \in F(X,E)$. Let $K$ be a compact subset of $X$ such that $f(X - K) = \{\lambda_1, \ldots, \lambda_n\}$. Since the support of $f$ is contained in the set $K \cup \bigcup\{Z(f - \lambda_i), 1 \leq i \leq n, \lambda_i \neq 0\}$, to complete the proof it suffices to see that $Z(f - \lambda_i)$ is compact for $\lambda_i \neq 0$. But this is true because, reasoning as in [GJ, p. 58], one deduces that, if $Z(f - \lambda_i)$ were not compact, then $f - \lambda_i$ would belong to some free maximal ideal $m$ of $C(X,E)$, which contradicts the fact $f \in m$.

**Remark.** Corollary 2 strengthens [D2, Corollary].

**Theorem 2.** If either $A = C^\#(X,E)$ or $A = F(X,E)$, then $M(C(X,E))$ $\rightarrow$ $M(A)$ given by $m \mapsto m \cap A$ is a homeomorphism.

**Proof.** In both cases $A$ is a subalgebra of $C(X,E)$ containing all the idempotents of $C(X,E)$, and $A$ is closed under inversion (i.e., if $f \in A$ and $Z(f) = \emptyset$, then $1/f \in A$). We shall see that the conclusion of the theorem is true for any such algebra.

First we shall show that every maximal ideal of $A$ is of type $m \cap A$ for some maximal ideal $m$ of $C(X,E)$. Let $M \in M(A)$ and $f_1, \ldots, f_n \in M$. If $\bigcap Z(f_i) = \emptyset$,
then there are idempotents $e_1, \ldots, e_n$ such that $\sum f_i e_i$ is a unit of $C(X, E)$ (see [D$_1$, Lemma]) and, since $A$ is closed under inversion, then $\sum f_i e_i$ is also a unit in $A$, which is contradictory to $\sum f_i e_i \in M$. So $\bigcap Z(f_i) \neq \emptyset$ and therefore there is a maximal ideal $m$ of $C(X, E)$ containing $M$. Then one has $M \subset m \cap A$ and, by the maximal character of $M$, one concludes that $M = m \cap A$.

Now let $m' \in M(C(X, E))$. Then $m' \cap A$ is a proper ideal of $A$, and hence there is $M \in M(A)$ such that $m' \cap A \subset M$. As we have just seen above, $M = m \cap A$ for some $m \in M(C(X, E))$, so $m' \cap A \subset m \cap A$. From this inclusion and the assumptions on $A$, it follows that $m = m'$ (see [D$_1$, Corollary to Proposition 2]). This shows that $m' \cap A$ is a maximal ideal of $A$. The same argument shows that the map $m \mapsto m \cap A$ is injective and, as for the maximal ideals of $C(X, E)$, two maximal ideals of $A$ containing the same idempotents agree. Hence finally one deduces that the map $m \mapsto m \cap A$ is an homeomorphism since it is a one-to-one and onto continuous map between two compact Hausdorff spaces.

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REFERENCES


[D$_3$] ———, Note on two subrings of $C(X)$, preprint.


