PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 3, July 1986

A TRANSFER FOR COMPACT LIE GROUP ACTIONS

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ABSTRACT. A short construction of a transfer homomorphism from $H_*(X/G)$ to $H_*(X/H)$, where $H \subseteq G$ are compact Lie groups and X is a G-CW-complex, is given.

We give here a quick construction of a transfer homomorphism for compact Lie group actions due to Lewis, May and McClure [**2**]. More specifically, we sketch a shortened version of their proof of

THEOREM. *Let G be a compact Lie group, $H \subseteq G$ a closed subgroup, and X a G-CW-complex. Then there exist homomorphisms (for any coefficient group R)*

$$\mathrm{trf}\colon H_*(X/G;R) \to H_*(X/H;R) \quad \textit{and}$$
$$\mathrm{trf}^*\colon H^*(X/H;R) \to H^*(X/G;R),$$

natural in X; and such that $\pi_* \circ \mathrm{trf}$ and $\mathrm{trf}^* \circ \pi^*$ are multiplication by $\chi(G/H)$. Here, $\pi\colon (X/H) \to X/G$ denotes the projection.

For convenience, we concentrate on the homology transfer—the construction in cohomology is the same. Let G be a fixed compact Lie group. By G-representation will always be meant a finite dimensional representation over $\mathbf{R}$. For any G-representation V, S^V will denote its one-point compactification. If X is a G-CW-complex with basepoint $x_0 \in X^G$, then we write $\Sigma^V X = S^V \wedge X$ for its V-suspension.

The following "skeletal approximation theorem" for suspensions will be needed.

LEMMA. *Let X, Y be pointed G-CW-complexes, and let V be a G-representation. Then any G-map $f\colon \Sigma^V X \to \Sigma^V Y$ is G-homotopic to a "skeletal" map f'; i.e., $f'(\Sigma^V(X^n)) \subseteq \Sigma^V(Y^n)$ for all $n \geq 0$.*

PROOF. For each n, $\Sigma^V(X^n)$ is constructed by attaching, to $\Sigma^V(X^{n-1})$, pairs of the form

$$(D(V) \times G/H \times D^n, (S(V) \times G/H \times D^n) \cup (D(V) \times G/H \times S^{n-1}))$$
$$\simeq (G/H \times D(V \times \mathbf{R}^n), G/H \times S(V \times \mathbf{R}^n)).$$

Here, $D(-)$ and $S(-)$ denote unit disk and sphere, and "attaching the pair (X_1, X_2)" means attaching X_1 via a map defined on X_2. Furthermore, for any G-space Z, there is a bijection

$$\mathrm{Map}_G(G/H \times D(V \times \mathbf{R}^n), Z) \simeq \mathrm{Map}_H(D(V \times \mathbf{R}^n), Z)$$

(note that $G/H \times D(V \times \mathbf{R}^n) \simeq G \times_H D(V \times \mathbf{R}^n)$).

Received by the editors July 25, 1984 and, in revised form, July 9, 1985.
1980 *Mathematics Subject Classification*. Primary 57S10; Secondary 54E60, 55N25.
Key words and phrases. Compact Lie group action, transfer homomorphism, orbit spaces, equivariant CW-complexes.

It thus suffices to prove, for any $H \subseteq G$, any $n \geq 0$, any H-equivariant map $\varphi: D(V \times \mathbf{R}^n) \to S^V \times Y$, and any H-homotopy $\Psi = (\psi_t)$ of $\psi = \varphi | S(V \times \mathbf{R}^n)$ with $\mathrm{Im}(\psi_1) \subseteq S^V \wedge Y^{n-1}$, that Ψ extends to a homotopy $\Phi = (\varphi_t)$ of φ such that $\mathrm{Im}(\varphi_1) \subseteq S^V \wedge Y^n$. For each $K \subseteq H$, set

$$d_K = \dim(V \times \mathbf{R}^n)^K = n + \dim(V^K).$$

If the pair $((S^V \wedge Y)^K, (S^V \wedge Y^n)^K)$ is d_K-connected for all $K \subseteq H$, then the construction of Φ is immediate. But $(S^V \wedge Y)^K$ is obtained by attaching, to $(S^V \wedge Y^n)^K$, pairs of the form

$$\begin{aligned}(D(V)^K \times (G/K_i)^K \times D^{n+j}, &(S(V)^K \times (G/K_i)^K \times D^{n+j}) \\ &\cup (D(V)^K \times (G/K_i)^K \times S^{n+j-1})) \\ &\cong (G/K_i)^K \times (D^{d_K+j}, S^{d_K+j-1})\end{aligned}$$

for $K_i \subseteq G$ and $j \geq 1$, so $((S^V \wedge Y)^K, (S^V \wedge Y^n)^K)$ is d_K-connected. □

The main step of the construction is the following "homology desuspension" result.

PROPOSITION. *Let V be a G-representation, X and Y (pointed) G-CW-complexes, and $f: \Sigma^V X \to \Sigma^V Y$ a G-map. Then there is an induced homomorphism (for any coefficient group R) $f_\#: \tilde{H}_*(X/G; R) \to \tilde{H}_*(Y/G; R)$, which is natural in f, and has the following properties:*

(1) $(\Sigma^W f)_\# = f_\#$ *for any G-representation W.*

(2) $f_\# = (f/G)_*$ *if $f: X \to Y$ (i.e., $V = 0$).*

(3) *If $\varphi: S^V \to S^V$ is a G-map of (ordinary) degree k, then $(\varphi \wedge \mathrm{id}_X)_\#$ is multiplication by k.*

PROOF. We may assume (suspending if necessary) that G acts orientably on V. Fix a map $f: \Sigma^V X \to \Sigma^V Y$. By the lemma, f is homotopic to a map f' such that $f'(\Sigma^V(X^n)) \subseteq \Sigma^V(Y^n)$ for all n.

Let $k = \dim(V)$. If

$$X^n = X^{n-1} \cup_\varphi \left(\coprod_i G/H_i \times D^n\right) \quad \text{and} \quad Y^n = Y^{n-1} \cup_\psi \left(\coprod_j G/K_j \times D^n\right)$$

(and X^{-1}, Y^{-1} are the basepoints), then

$$H_{n+k}(\Sigma^V(X^n)), \Sigma^V(X^{n-1}); R) \cong H_n(X^n, X^{n-1}; R) \cong \sum_i H_0(G/H_i; R)$$

and

$$H_{n+k}(\Sigma^V(Y^n), \Sigma^V(Y^{n-1}); R) \cong H_n(Y^n, Y^{n-1}; R) \cong \sum_j H_0(G/K_j; R).$$

Hence, dividing out linearly with the action of $\overline{G} = \pi_0(G)$, we can identify

$$\begin{aligned}R \otimes_{R[\overline{G}]} H_{n+k}(\Sigma^V(X^n), \Sigma^V(X^{n-1})) &\cong \sum_i R \cong H_n(X^n/G, X^{n-1}/G; R) \\ &= C_n(X/G; R)\end{aligned}$$

and similarly for Y. So f' induces homomorphisms (for all $n \geq 0$)

$$\hat{f}'_n = R \otimes_{R[\overline{G}]} H_{n+k}(f'_n, f'_{n-1}) \colon C_n(X/G; R) \to C_n(Y/G; R).$$

These clearly commute with the boundary maps and hence induce

$$f_\# = f'_\# \colon \tilde{H}_n(X/G; R) \to \tilde{H}_n(Y/G; R).$$

If f'' is another skeletal approximation to f, then let F be a skeleton preserving homotopy from f' to f''. Then $\hat{F}_n$ (in the above notation) defines a chain homotopy between $\hat{f}'_n$ and $\hat{f}''_n$. It follows that $f_\#$ is uniquely defined independently of the choice of f'.

Naturality of $f_\#$ and conditions (1) and (2) are now clear. If $\varphi \colon S^V \to S^V$ is any G-map, then

$$(\varphi \wedge \mathrm{id}_X)_* \colon H_{n+k}(\Sigma^V(X^n), \Sigma^V(X^{n-1})) \to H_{n+k}(\Sigma^V(X^n), \Sigma^V(X^{n-1}))$$

is multiplication by $\deg(\varphi)$ for any $n \geq 0$; and hence $(\varphi \wedge \mathrm{id}_X)_\#$ is multiplication by $\deg(\varphi)$. $\square$

The construction of the transfer now follows easily. Fix $H \subseteq G$, and let V be any G-representation having an embedding $G/H \subseteq V$. Let t denote the composite

$$t \colon S^V \to T(\nu_{G/H}) \subseteq T(\nu_{G/H} \oplus \tau_{G/H}) = G/H^+ \wedge S^V,$$

where $T(\nu_{G/H})$ is the Thom space of the normal bundle, and the first map the Pontrjagin-Thom construction. Note that the composite

$$S^V \xrightarrow{t} G/H^+ \wedge S^V \xrightarrow{\text{proj}} S^V$$

has degree $\chi(G/H)$ (see [**1**, Theorem 2.4]).

If X is any (nonbased) G-CW-complex, set

$$t_X = t \wedge \mathrm{id}_X \colon \Sigma^V(X^+) \to G/H^+ \wedge S^V \wedge X^+ \cong \Sigma^V(G/H \times X)^+.$$

Since $G/H \times X$ has the G-homotopy type of a G-CW-complex (Lemma 1) and $(G/H \times X)/G \cong X/H$, we can now define the transfer for X by

$$\mathrm{trf} = (t_X)_\# \colon H_*(X/G; R) \to H_*((G/H \times X)/G; R) = H_*(X/H; R).$$

Naturality of trf follows from the naturality of the homology desuspension. That $\pi_* \circ \mathrm{trf}$ is multiplication by $\chi(G/H)$ follows from (3) in the proposition.

References

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