ON A CONNECTED DENSE PROPER SUBGROUP OF $R^2$
WHOSE COMPLEMENT IS CONNECTED

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Abstract. A simple proof is given to the theorem of F. B. Jones which asserts the existence of such a subgroup of the additive group $R^2$ as in the title.

A real-valued function $f: R \to R$ is said to be additive if it satisfies

$$f(x + y) = f(x) + f(y) \quad (x, y \in R).$$

If we regard $R$ as a vector space over the rational numbers $Q$, then $f$ is additive if and only if $f$ is $Q$-linear. The graph $G_f = \{(x, f(x)): x \in R\}$ of an additive function $f$ is a subgroup of $R^2$. The purpose of this note is to give a simple proof to the following theorem of F. B. Jones (Theorem 5 and Property 1 in [1]).

Theorem. There exists an additive function $f: R \to R$ such that both the graph $G_f$ and its complement $G_f^c$ are connected and dense in the plane $R^2$.

Let $p$ denote the perpendicular projection of the plane $R^2$ onto the $x$-axis $R$ ($= R \times 0$). We say that a set $M$ in $R^2$ has a positive width if $p(M)$ contains an open interval of the $x$-axis. We denote by $\Omega$ the collection of all closed subsets of $R^2$ which have a positive width. Then the preceding theorem is a consequence of the following two propositions:

Proposition A. There exist an additive function $f: R \to R$ such that its graph $G_f$ intersects every member $M$ of $\Omega$.

Proposition B. If a graph $G_h$ intersects every member $M$ of $\Omega$, then both $G_h$ and $G_h^c$ are dense connected subsets of $R^2$.

To prove the propositions, we begin with the following

Lemma. $\Omega$ and $R$ have the same cardinality: $|\Omega| = |R|$.

Proof. Every straight line parallel to the $x$-axis is a member of $\Omega$. Hence $|R| \leq |Q|$. Let $[U_1, U_2, U_3, \ldots]$ be a countable basis for the topology of $R^2$. Every open set $G$ of $R^2$ is the union of all $U_n$ which are contained in $G$. For each $M \in \Omega$, we define

$$s(M) = \{n: U_n \subset M^c\}.$$
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Then $s : \Omega \to 2^N$ is injective, where $N$ is the set of all positive integers. Hence, $|\Omega| \leq |2^N| = |R|$ and therefore, $|\Omega| = |R|$.

**Proof of Proposition A.** Let $X = \{x_j : j \in J\}$ be a $Q$-basis (i.e., a Hamel basis) of vector space $R$ over $Q$. Then, $|X| \leq |R| \leq |X| \times |Q| = |X|$, and hence $|X| = |R|$. So there is a one-to-one correspondence $x_j \leftrightarrow M_j$ ($j \in J$) between $X$ and $\Omega$. 

For each $j \in J$, since $p(M_j)$ contains an open interval and since $\{q \cdot x_j : q \in Q\}$ is dense in $R$, there exists a $q_j \in Q$ ($q_j \neq 0$) such that $q_j x_j \in p(M_j)$. Hence, we can find $y_j \in R$ such that $(q_j x_j, y_j) \in M_j$. Let $x'_j = q_j x_j$. Then $X' = \{x'_j : j \in J\}$ is also a $Q$-basis. Define $f_j : X_1 \to R$ by $f_j(x'_j) = y_j$ ($j \in J$), and extend $f_1$ to a $Q$-linear function $f : R \to R$. Then $f$ satisfies the required condition.

**Proof of Proposition B.** Every nonempty open set in the plane contains a member $M$ of $\Omega$, and $M$ contains a point of $G_h$. Hence, $G_h$ is dense. Let $h(x) = h(x) + 1$ ($x \in R$). Then graph $G_h$ is a translation of $G_h$ and hence dense in $R^2$. Clearly $G_h \subset G_h^c$ and so $G_h^c$ is also dense. By the fact $G_h \subset G_h^c \subset G_h^c (= R^2)$, we know that if $G_h$ is connected then so is $G_h^c$. Since $G_h$ is homeomorphic to $G_h$, the only one we need to show that $G_h$ is connected.

Suppose that $G_h$ is the sum of mutually separated nonempty sets $H$ and $K$:

$$\quad G_h = H \cup K, \quad \overline{H} \cap K = 0 = H \cap \overline{K}, \quad H \neq 0 \neq K.$$ 

Let $M = \overline{H} \cap \overline{K}$. Then $M$ separates the plane $R^2$. In fact,

$$R^2 - M = (\overline{H} \cap \overline{K})^c = \overline{H}^c \cup \overline{K}^c,$$

$$\overline{H}^c \cap \overline{K}^c = (\overline{H} \cup \overline{K})^c = G_h^c = 0,$$

$$\overline{H}^c \supset K \neq 0, \quad \text{and similarly} \quad \overline{K}^c \neq 0.$$ 

$M$ ($\subset \overline{K}, \overline{H}$) intersects neither $H$ nor $K$, and hence $G_h \cap M = 0$. By the assumption, $M$ does not have a positive width. Thus the complement of $p(M)$, say $A$, in the $x$-axis is dense. The open set $R^2 - M$ contains graph $G_h$ and every line $\pi^{-1}(a)$ for $a \in A$. This is a contradiction to the following lemma:

**Lemma.** Let $A$ be a dense subset of the $x$-axis $R$. Suppose that an open subset $W$ of the plane satisfies the conditions:

(i) $W \cap \pi^{-1}(x) \neq 0$ for all $x \in R$, and

(ii) $\pi^{-1}(a) \subset W$ for all $a \in A$.

Then $W$ is path-connected.

**Proof.** Let $a_0 \in A$ and let $W_0$ be the path component of $W$ which contains $a_0$. For any $z \in W$, we can find a path-connected neighborhood $U(z)$ of $z$ which is contained in $W$. Since $A$ is dense in the $x$-axis, $U(z)$ meets a line $\pi^{-1}(a)$ for some $a \in A$. Hence, $z$ and $a$ can be joined by a path in the subset $U(z) \cup \pi^{-1}(a)$ of $W$.

Now we prove that $a \in W_0$. Suppose $a \in W - W_0$. We may assume $a_0 < a$. Let $A_1 = \{b \in A : a_0 < b, \ b \in W - W_0\}$, and let $\xi = \inf A_1$. Since $W_0$ is an open set containing $a_0$, we have $a_0 < \xi$. By (i), there is a point $z_1 \in W \cap \pi^{-1}(\xi)$. Let $U(z_1)$ be a path-connected neighborhood of $z_1$ contained in $W$. Then $\pi(U(z_1))$ is a
neighborhood (in the x-axis) of $\xi$, and contains a point $b' \in A_1$ and a point $a' \in A$ such that $a_0 \leq a' < \xi \leq b'$. By the definition of $\xi$, $a' \in W_0$. Points $a'$ and $b'$ are in $p^{-1}(a') \cup U(z_1) \cup p^{-1}(b')$ which is a path-connected subset of $W$. Hence, $b' \in W_0$. This contradicts that $b' \in A_1$, and completes the proof.

References

1. F. B. Jones, Connected and disconnected plane sets and the functional equation $f(x) + f(y) = f(x + y)$, Bull. Amer. Math. Soc. 48 (1942), 115–120.

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