ON STABLY EXTENDED PROJECTIVE MODULES
OVER POLYNOMIAL RINGS

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ABSTRACT. We prove here that if \( A \) is a commutative noetherian ring of
Krull dimension \( d \) and of finite characteristic prime to \( d! \), then stably extended
projective \( A[X_1, \ldots, X_n] \)-modules of rank \( \geq d/2 + 1 \) are extended from \( A \).

We denote by \( A \) a commutative ring with unit. \( U_r(A) \) is the set of unimodular
rows of length \( r \) over \( A \). As in [3, §5, p. 34], given \( u, v \) in \( U_r(A) \) and a subgroup \( G \)
of \( GL_r(A) \), we write \( u \sim_G v \) if there exists \( g \) in \( G \) such that \( v = ug \). We abbreviate
the notations \( u \sim_{GL_r(A)} v \) to \( u \sim v \) and \( u \sim_{E_r(A)} v \) to \( u \sim_E v \). For \( u, v \) in \( U_r(A) \)
we denote by \( u \leftrightarrow_{GL_r(A)} v \) or simply \( u \leftrightarrow v \) the property: \( u \sim (1, 0, \ldots, 0) \) if and
only if \( v \sim (1, 0, \ldots, 0) \).

If \( \varphi: A \rightarrow B \) is a canonical ring homomorphism (such as \( A \rightarrow A_S, A \rightarrow A/I \),
where \( S \) is a multiplicative subset and \( I \) is an ideal of \( A \)) and \( a \in A \), we denote
\( \varphi(a) = \bar{a} \).

If \( f(X) \) is a polynomial in \( A[X] \), we denote its leading coefficient by \( l(f) \). As
usual \( A(X) \) denotes the localization of \( A[X] \) at the set of monic polynomials. If
\( S \) is a multiplicative subset of \( A \) and \( f(X) \in A[X] \), we say that \( f(X) \) is unitary
in \( A_S[X] \) if \( f(X) \) is unitary in \( A_S[X] \), that is, \( l(f) = us \) for some \( s \in S \) and \( u \)
invertible in \( A \).

We recall that a finitely generated projective module \( P \) over \( R = A[X_1, \ldots, X_n] \)
is called stably extended from \( A \) if there exists a finitely generated \( R \)-projective
module \( Q \) extended from \( A \) such that \( P \oplus Q \) is extended from \( A \) or, equivalently, if
there exists \( m \geq 0 \) such that \( P \oplus R^m \) is extended from \( A \).

**Lemma 1** (cf. [12, Corollary 2]). Let \( (x_0, \ldots, x_r) \in U_{r+1}(A), r \geq 2, \)
and let \( t \) be an element of \( A \) which is invertible mod \((Ax_0 + \cdots + Ax_{r-2})\). Then
\( (x_0, \ldots, x_r) \sim_E (x_0, \ldots, t^2x_r) \).

**Proof.** Let \( \sum_{i=0}^r x_iy_i = 1 \). Then by [8, Lemma 1] we have
\[
(x_0, \ldots, x_r-2, x_{r-1}, x_r) \sim_E (x_0, \ldots, x_r-2, y_{r-1}, y_r) \sim_E (x_0, \ldots, x_r-2, tx_{r-1}, tx_r).
\]
Let \( tt' \equiv 1 \) mod \((Ax_0 + \cdots + Ax_{r-2})\). By Whitehead’s lemma we have
\[
(x_0, \ldots, x_r-2, tx_{r-1}, tx_r) \sim_E (x_0, \ldots, x_r-2, t'tx_{r-1}, t^2x_r) \sim_E (x_0, \ldots, x_r-2, x_{r-1}, t^2x_r). \quad \square
\]

**Lemma 2** (cf. E.G., [1, Théorème 1]). Let \( S \) be a multiplicative subset
of \( A \), such that \( A_S \) is noetherian of finite Krull dimension \( d \). Let \( (\bar{a}_0, \ldots, \bar{a}_r) \in \)

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$U_{r+1}(A_S),\ r>d$. Then there exist $b_i \in A\ (1 \leq i \leq r)$ and $s \in S$ such that $s \in A'(a_1 + b_1a_0) + \cdots + A'(a_r + b_ra_0)$.

**Proof.** Similar to that of [1, Théorème 1, §3]. We have to choose elements in $A$ (as $a'_i$ in [1, §3, Lemma 2]) in order to avoid certain prime ideals in $A$ which come from prime ideals in $A_S$. Finally we obtain $A_S(a_1 + b_1a_0) + \cdots + A_S(a_r + b_ra_0) = A_S$. □

**Lemma 3.** Let $f(X)$ be a polynomial in $R = A[X]$ of degree $n > 0$, such that $f(0)$ is invertible in $A$. Then for any $g(X) \in A[X]$ and natural $k \geq \deg g(X) - 1$ there exists $h_k(X) \in A[X]$ of degree $< n$ such that $g(X) = X^k h_k(X) \mod (Rf(X))$.

**Proof.** Let $f(X) = a_0 + \cdots + a_nX^n,\ g(X) = c_0 + \cdots + c_mX^m$. Let $g(X) - c_0a_0^{-1}f(X) = Xh_1(X)$. Then $g(X) \equiv Xh_1(X) \mod (Rf(X))$ and $\deg h_1(X) < \max(m, n)$. Similarly we obtain $h_2(X)$ such that $h_1(X) \equiv Xh_2(X) \mod (Rf(X)),\ g(X) = X^2h_2(X) \mod (Rf(X)),\ \deg h_2(X) < \max(m - 1, n)$, etc. In this way the lemma easily follows. □

**Lemma 4.** Let $(x_0, \ldots, x_r) \in U_{r+1}(A)$ and $k \equiv 1 \mod (r!)$. Then for any $0 \leq i \leq r$ we have $(x_0, \ldots, x_i, \ldots, x_r) \leftrightarrow (x_0, \ldots, x^k, \ldots, x_r)$.

**Proof.** Let $i = 0$. Any unimodular row which contains a $(k - 1)$-power of an element in $A$ is completable to an invertible matrix by [9, Theorem 2]. It follows by [9, Corollary 3.3], that if $(x_0, \ldots, x_r) \sim (1, 0, \ldots, 0)$, then $(x_0^k, \ldots, x_r) \sim (1, 0, \ldots, 0)$. On the other hand if $(x_0^k, \ldots, x_r) \sim (1, 0, \ldots, 0)$, then let $x_0x_0^k \equiv 1 \mod (Ax_1 + \cdots + Ax_r)$ and so,

$$(x_0, \ldots, x_r) \sim E (x_0^kx_0^{k-1}, \ldots, x_r) \sim (1, 0, \ldots, 0)$$

by [9, Theorem 2] and [9, Corollary 3.3]. □

**Theorem 5 (cf. [1, Théorème 3]).** Let $A$ be a commutative noetherian ring of finite Krull dimension $d$, let $r > d/2 + 1$ and assume that $A$ is of finite characteristic prime to $r!$. Let $P$ be a finitely generated projective module of rank $r$ over $R = A[X_1, \ldots, X_n]$ such that $P \otimes R$ is extended from $A$. Then $P$ is extended from $A$.

**Proof.** By Quillen’s Patching Theorem [6, Theorem 1’ or 3, Chapter 5, §1] we may assume $A$ to be local, so char $A$ is a power of a prime $p$. Let $P_0$ be the $A$-module $P/(X_1P + \cdots + XnP)$. We have to show $P \cong P_0 \otimes_R R$. As $p$ is in the nilradical of $R$, it is enough to show $P/pP \cong (P_0 \otimes_R R)/p(P_0 \otimes_R R)$ (see e.g. [3, Chapter 1, Corollary 1.6]). We have

$$\frac{P_0 \otimes_R R}{p(P_0 \otimes_R R)} \cong \frac{P_0}{pP_0} \otimes_{A/pA} \frac{R}{pR} = \frac{A}{pA}[X_1, \ldots, X_n],$$

which means that the $R/pR$-module $(P_0 \otimes_R R)/p(P_0 \otimes_R R)$ is extended from the $A/pA$-module $P_0/pP_0$. Therefore we have to show that the $R/pR$-module $P/pP$ is extended from $A/pA$. Replacing $A$ by $A/pA$, we assume char $A = p$. By the Quillen induction (see [6 or 3, Chapter 5, §3]) we reduce to the case $n = 1, R = A[X]$: Let $n > 1$ and let $S$ be the set of monic polynomials in $A[X_1], A(X_1) = A[X_1]$. Then $\dim A(X_1) = \dim A$ [3, Chapter 4, Proposition 1.2] and so by induction the
A(XX)-module $P$ is extended from $A(XX)$. By Horrocks’ theorem (see e.g. [3, Chapter 4]), $P$ is extended from $A$.

We have to prove that $GL_{r+1}(R)$ acts transitively on $U_{r+1}(R)$. Let us call admissible transformations of a row $u \in U_{r+1}(R)$ elementary transformations and also transformations of the type

$$(x_0, \ldots, x_i, \ldots, x_r) \rightarrow (x_0, \ldots, x_i^k, \ldots, x_r), \quad \text{where } k \equiv 1 \mod (r!).$$

By Lemma 4 it is enough to prove the following

Claim. If $u(X) = (f_0(X), \ldots, f_{r+1}(X)) \in U_{r+1}(R)$, $r \geq 2$, then $u$ can be transformed to $(1,0,\ldots,0)$ using admissible transformations.

We prove the claim by induction on the number $N$ of nonzero coefficients of the polynomials $f_0(X), \ldots, f_r(X)$, starting with $N = 1$. Let $N > 1$. We may assume $\deg f_0 > 0$. Let $l(f_0) = a$. If $a$ is invertible, then $u \sim (1,0,\ldots,0)$ (see, e.g., [2, Chapter III, Corollary 1.4]). In our case the proof is much simpler: As $f_0(X)$ is unitary and $A$ is local, there exist just a finite number of maximal ideals in $R$ which contain $f_0$, so there exists $g$ in $R$ which does not belong to any such ideal and $f_1 \equiv g \mod (Rf_2 + \cdots + Rf_r)$. As $Rf_0 + Rg = R$, we conclude $u \sim (1,0,\ldots,0)$. We assume now that $a$ is not invertible in $A$. By the inductive assumption with respect to the ring $\bar{A} = A/aA$ and the row $\bar{u}(X)$, we can obtain from $u(X)$ a row $\bar{v}(X) \equiv (1,0,\ldots,0) \mod (Ra)$ using admissible transformations over $R$. We can perform such transformations so that at every stage the row contains a polynomial which is unitary in $R_a$. Indeed, if we have to perform, e.g., the elementary transformation

$$(g_0, \ldots, g_r) \rightarrow (g_0 + aXmg_1, g_1, \ldots, g_r)$$

and $g_1$ is unitary in $R_a$, then we replace $T$ by the following two transformations:

$$(g_0, g_1, \ldots, g_r) \rightarrow (g_0 + aX^m g_1, g_1, \ldots, g_r, g_0 + aX^m g_1, \ldots, g_r),$$

where $m > \deg g_0$. We assume now $(f_0,\ldots, f_r) \equiv (1,0,\ldots,0) \mod (Ra)$ and $f_i$ is unitary in $R_a$. If $i > 0$, then replace $f_0$ by $f_0 + aX^m f_i$, where $m > \deg f_0$; so we assume $f_0$ is unitary in $R_a$ and also $\deg f_0 > 0$. By Lemma 3 we assume now $f_i = X^{2k} g_i$, where $\deg g_i < \deg f_i$ for $1 \leq i \leq r$. By Lemma 1 we assume $\deg f_i < \deg f_0$ for $1 \leq i \leq r$.

Let $\deg f_0 = m_0$. If $m_0 = 1$, then $f_i \in A$ for $1 \leq i \leq r$. Therefore for sufficiently big $q$ we have $(f_0(X) - f_0(0))^q \in Rf_1 + \cdots + Rf_r$. Choose such $q$ of the form $p^n$ and $q \equiv 1 \mod (r!)$. Then we perform the admissible transformation $(f_0,\ldots, f_r) \rightarrow (f_0^q,\ldots, f_r)$. As char $A = p$, we have $f_0^q = f_0(0) + (f_0(X) - f_0(0))^q$, so $(f_0^q,\ldots, f_r) \sim (Ra)^{\deg f_0} (1,0,\ldots,0)$.

Assume now $m_0 \geq 2$. We use an argument similar to that in the proof of [1, §4, Theorem 3']. Let $(c_1,\ldots, c_{m_0(r-1)})$ be the coefficients of $1, X, \ldots, X^{m_0-1}$ in the polynomials $f_2(X),\ldots, f_r(X)$. By [3, Chapter III, Lemma 1.1], the ideal generated in $A_a$ by $A_a \cap (R_a f_0 + R_a f_1)$ and the coefficients of $f_i^q$ ($2 \leq i \leq r$) is $A_a$. As $m_0(r-1) \geq 2 \cdot d/2 = d > \dim R_a$, by Lemma 2 there exists $(c_1',\ldots, c_{m_0(r-1)}) \equiv (c_1,\ldots, c_{m_0(r-1)}) \mod ((Rf_0 + Rf_1) \cap A)$, such that $A_a c'_1 + \cdots + A_a c'_{m_0(r-1)} = A_a$. Assume that we have already $A_a c_1 + \cdots + A_a c_{m_0(r-1)} = A_a$. By [1, §4, Lemma 1(b)], the ideal $A_f_0 + A f_2 + \cdots + A f_r$ contains a polynomial $h(X)$ of degree $m_0 - 1$.
which is unitary in \( R_a \). Let \( l(h) = u a^k \), where \( u \) is invertible in \( A \). Using Lemma 1, we achieve by elementary transformations

\[
(f_0, f_1, \ldots, f_r) \rightarrow (f_0, a^{2k} f_1, \ldots, f_r)
\]

\[
\rightarrow (f_0, a^{2k} f_1 + (1 - a^k u^{-1} l(f_1) h, \ldots, f_r)).
\]

Now, \( a^{2k} f_1 + (1 - a^k u^{-1} l(f_1)) h \) is unitary in \( R_a \), so assume \( f_1 \) is unitary in \( R_a \), \( \deg f_1 = m_1 < \deg f_0 \). By Lemma 1 we assume also \( \deg f_i < m_1 \) for \( 2 \leq i \leq r \).

Repeating the argument above we lower the degree of \( f_1 \) and obtain finally a row of the form \((f_0, f_1, \ldots, f_r) \equiv (1, 0, \ldots, 0) \) mod \((R_a)\) with \( f_0 \) unitary in \( R_a \), \( m_0 = \deg f_0 > \deg f_1 = 1 \) and \( f_i \in A \) for \( 2 \leq i \leq r \). Let \( l(f_1) = u a^k \), where \( u \) is invertible in \( A \), \( f_0(X) = 1 - a g(X) \).

We have by Lemma 3

\[
f_2 \equiv f_2 + 1 - a^{km_0} g^{km_0} \equiv f_2 + 1 - a^{km_0} X^q h(X) \mod (R_{f_0})
\]

for some \( q \) of the form \( p^n \), \( q \equiv 1 \mod (r!) \) and \( h(X) \) of degree \( < m_0 \). As \( \deg h(X) < m_0 \), we have \( a^{km_0} h(X) \equiv b \mod (R_{f_1}) \) for some \( b \in A \), so we can obtain \( f_2(X) \) of the form \( f_2(X) = u + c X^q \) with \( c \in A \), \( u \) invertible in \( A \). By admissible transformations \((f_0, f_1, \ldots, f_r) \rightarrow (f_0, f'_1, f'_r, \ldots, f_r)\) we obtain \( f_0, f'_1 \in [A[X]] \). By Lemma 3 we obtain a row of the type \((c_0, c_1, f_2, c_3, \ldots, c_r)\), where \( c_i \in A \), so by the argument in the case \( m_0 = 1 \) we finish the proof. \( \square \)

Using [11, Theorem 1.1] we obtain

**Corollary 6.** If \( A \) is a noetherian ring of dimension \( d \) and of characteristic prime to \( d! \), then projective stably extended \( A[X_1, \ldots, X_n] \)-modules of rank \( \geq d/2 + 1 \) are extended from \( A \).

**Corollary 7.** If \( A \) is a noetherian ring of dimension \( 2 \) and of finite odd characteristic, then projective stably extended \( A[X_1, \ldots, X_n] \)-modules are extended from \( A \) (that is, \( A[X_1, \ldots, X_n] \) is a Hermite ring).

Finally, we obtain the following particular case of the Bass-Quillen conjecture:

**Corollary 8.** If \( A \) is a noetherian regular ring of dimension \( d \) and of characteristic prime to \( d! \), then finitely generated projective \( A[X_1, \ldots, X_n] \)-modules of rank \( \geq d/2 + 1 \) are extended from \( A \).

Without the assumption on the characteristic, Corollary 6 would strengthen Theorem 1.1 in [11] restricted to polynomial rings (see also [10, Theorem 7.2]) and Corollaries 7 and 8 would generalize the Murthy-Horrocks Theorem (see, e.g., [3, Chapter V, Theorem 3.3]).

It can be shown using [10, Theorem 7.2] and the arguments above that if \( A \) is noetherian of dimension \( d \) and \( u \in U_{d+1}(A[X]) \), then there exist \( u_1, u_2 \) in \( U_{d+1}(A[X]) \) such that \( u \sim_E u_1 \sim_E u_2 \), \( u_1 \) is of the form \((1 + a X^n, b + a X, c_2, \ldots, c_r)\) with \( a, b, c_2, \ldots, c_r \) in \( A \) and \( u_2 \) is of the form \((1 + a X^n, 1 + b X^m, c_2, \ldots, c_r)\) with \( a, b, c_2, \ldots, c_r \) in \( A \).

Finally we mention some related results. If we assume \( A \) is regular, then projective \( A[X_1, \ldots, X_n] \)-modules are extended under certain additional assumptions not necessarily related to Krull dimension; e.g., by Lindel's theorem [4] this holds if \( A \) is a regular algebra of essentially finite type over a field (see also [5]). For a survey of further results apart from those in [3] see [13]. For more recent results see Suslin's work in Trudy Mat. Inst. Steklov. 168 (1984) (English translation to appear in Proc. Steklov Inst. Math.). See also [7].
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