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THE THOMPSON-WIELANDT THEOREM

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ABSTRACT. In this short paper we give a completely elementary argument for the Thompson-Wielandt theorem and note its significance for the Goldschmidt-Sims conjecture.

A recent fundamental paper of Goldschmidt [**Gs1**] has renewed interest in the study of amalgams of finite groups. The underlying problem of the current works on amalgams is the so-called Goldschmidt-Sims problem. A theoretical starting point for this is the Thompson-Wielandt theorem [**T**]. This asserts that every "nontrivial" primitive amalgam must be of characteristic p-type and that the common group in the amalgam is "almost" a p-group, for some prime p. Here, we offer a transparent argument of how the Thompson-Wielandt theorem is a direct consequence of primitivity. Please consult [**Gs1**] for the definitions not given below.

We denote an amalgam by $P_1 \supseteq B \subseteq P_2$, where P_1, P_2 and B are the obvious subgroups of $P_1 *_B P_2$. It is said to be *primitive* if, for $i = 1, 2$, whenever $1 \neq K \trianglelefteq P_i$ and $K \subseteq B$, we have $N_{P_{3-i}}(K) = B$. A justification for *primitive* amalgams is that any two distinct maximal subgroups of a finite simple group form such an amalgam. Further, the completion graph of the amalgam given by two minimal parabolics of a rank 2 Lie-type group is the building associated to that group. The rather ambitious Goldschmidt-Sims problem is then to *classify all possible primitive amalgams.* Prompted by possible applications, many special cases of the above have been done. Among these are the works of Goldschmidt [**Gs1**], Delgado [**D**], Fan [**F**], Stellmacher [**St1**], and Stroth [**Str1, Str2**].

It was Goldschmidt who first fully and successfully exploited the underlying geometry of amalgams. Take $P_1 \supseteq B \subseteq P_2$ to be an arbitrary primitive amalgam, set $G = P_1 *_B P_2$, and let Γ be the usual coset graph, i.e., the completion graph $\Gamma(G, P_1, P_2)$ (see [**Gs1**]). It is then well known that

(1) *Γ is a tree whose valences at adjacent vertices are $|P_1\colon B|$ and $|P_2\colon B|$.*

(2) *G is an edge transitive group of automorphisms of Γ.*

(3) *For adjacent vertices α and β in Γ, $G_\alpha \supseteq G_{\alpha,\beta} \subseteq G_\beta$ is isomorphic to $P_1 \supseteq B \subseteq P_2$.*

It has been usually useful to investigate a baby version of the Goldschmidt-Sims problem, which we refer to as the Goldschmidt-Sims conjecture:

> *There exists an integer constant M such that for any G and Γ as described above, no nontrivial element of G fixes all the vertices with distance M from some vertex α in Γ.*

Under an additional stipulation, Goldschmidt [**Gs2**] has obtained a (nongeometric) bound on the number of possible primitive amalgams with a given index. Of

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course, one of the points of the Thompson-Wielandt theorem is that it suffices to do the above problem for $O_p(G_\alpha)$, where p is some prime.

Henceforth let us take G to be just a group of automorphisms of a graph Γ and fix two adjacent vertices 1 and 2 in Γ with finite valence at each vertex. For $i \in \{1, 2\}$, set

$$\Delta^1(i) = \{(3-i)^x \colon x \in G_i\} \cup \{i\}$$

and inductively define

$$\Delta^n(i) = \bigcup_{x \in G_i} \Delta^{n-1}(3-i)^x.$$

Take G_i^n to be the set of all elements in G that fix $\Delta^n(i)$ elementwise.

THEOREM (THOMPSON-WIELANDT). *Let G and Γ be as above. Suppose $G_1 \supseteq G_{1,2} \subseteq G_2$ is a primitive amalgam and G_1^2 and G_2^2 are nontrivial finite groups. Then for some prime p, $F^*(G_1)$, $F^*(G_2)$ and $F^*(G_{1,2})$ and either G_1^2 or G_2^2 are p-groups.*

We first give a proof of the following known fact to demonstrate that our argument is completely elementary.

LEMMA. *If $H \lhd\lhd X$, then we have that*
(1) *$E(X)$ normalizes H.*
(2) *$O_p(X)$ normalizes $O^p(H)$.*

PROOF. For (2) we simply set $X_0 = O_p(X)O^p(H)$ and notice that since $O^p(H) \lhd\lhd X_0$, it must be the case that $O^p(X_0) = O^p(H)$.

To prove (1), it suffices to show that for every component K of X and $U \trianglelefteq X$, we have either $[U, K] = 1$ or $K \subseteq U$. Since we may take $K \neq X$, it follows as a consequence of induction that either $[[U, K]K] = 1$ or $K \subseteq [U, K]$. This completes our proof. $\square$

PROOF OF THEOREM. Take q to be any prime with $O^q(G_i^2) \neq 1$ for some $i \in \{1, 2\}$. Since for all $i' \in \Delta^1(3-i)$, we have $G_{i'}^2 \trianglelefteq G_{3-i}^1 \trianglelefteq \langle G_{1,2}, G_{3-i}\rangle$, it follows from our lemma and primitivity that $O_q(G_{1,2})E(G_{1,2}) \cup O_q(G_{3-i})E(G_{3-i}) \subseteq G_{3-i,i'}$. Therefore,

$$O_q(G_{1,2})E(G_{1,2}) = O_q(G_{3-i}^1)E(G_{3-i}^1) = O_q(G_{3-i})E(G_{3-i}).$$

Consequently, it is now apparent that $1 = E(G_{1,2}) = E(G_1) = E(G_2)$ and that there must be a prime p for which, say, $O^p(G_1^2) = 1$.

Let us now take q to be any prime not equal to p. We then claim $O^q(G_2^2) \neq 1$. Since $O^q(G_1^2) \neq 1$, we obtain from the above observation that $O_q(G_2^2) \subseteq O_q(G_1^1) \subseteq G_2^1$. But this means $O_q(G_1^1) \subseteq \bigcap_{x \in G_1}(G_2^1)^x = G_1^2$, which yields our claim. Now it is immediate that $O_q(G_1) = O_q(G_{1,2}) = O_q(G_2) = 1$. $\square$

Let us say that our amalgam $P_1 \supseteq B \subseteq P_2$ is of *characteristic p-type* if $F^*(P_1)$, $F^*(P_2)$ and $F^*(B)$ are p-grouups, where p is a prime. In light of the above theorem, the hypothesis for the Goldschmidt-Sims conjecture may be taken to be

G is an edge transitive group of automorphisms of a tree Γ such that $G_\alpha \supseteq G_{\alpha,\beta} \subseteq G_\beta$, where $\alpha \sim \beta$ in Γ, is a characteristic p-type primitive amalgam of finite groups for some prime p.

We then define $Z_\delta = \langle Z(P) \colon P \in \mathrm{Syl}_p(G_\delta)\rangle$, where δ is a vertex in Γ. Set b_δ to be the maximal integer such that $Z_\delta \subseteq G_\delta^{b_\delta}$ and let $b = \min(b_\alpha, b_\beta)$. Under the additional assumption that G_δ/G_δ^1 is a rank 1 Lie-type group, it is now known what the isomorphism classes of G_α and G_β must be through the works of [**Gs1, D, F, St1, Str1** and **Str2**]. Furthermore, Delgado and Stellmacher have recently produced a uniform treatment of the above in which the major part of the work involves the precise determination of the parameter b (see [**St2**]). Thus it seems one may want to obtain b under the general hypothesis of the Goldschmidt-Sims problem. We observe below that this is actually equivalent to resolving the Goldschmidt-Sims conjecture.

Let us take m_δ to be the maximal integer such that $G_\delta^{m_\delta} \neq 1$ and set $m = \min(m_\alpha, m_\beta)$. Of course, $|m_\alpha - m_\beta| \leq 1$.

LEMMA. *Suppose G and Γ satisfy the hypothesis of the Goldschmidt-Sims conjecture and $m \geq 2$. Then $b = m - 1$ or m.*

PROOF. We see that it suffices to show for any vertex $\delta \in \Gamma$ that $Z_\delta \subseteq G_\delta^{m-1}$. So take any path $(\delta, \delta+1, \ldots, \delta+k)$ of length $k \leq m-1$. Then $G_\delta^1 \triangleright G_{\delta+1}^2 \triangleright \cdots \triangleright O_p(G_{\delta+k}^{m_{\delta+k}})$. Hence, by induction and primitivity we get $Z_\delta \subseteq G_{\delta+k}$. $\square$

REMARK. A lemma of Bender [**G**, p. 43], similar in spirit to the Thompson-Wielandt theorem, states that if $G_1 \supseteq G_{1,2} \subseteq G_2$ is a primitive amalgam and $F^*(G_1), F^*(G_2) \subseteq G_{1,2}$, then $F^*(G_1)$, $F^*(G_2)$ and $F^*(G_{1,2})$ are p-groups for some prime p. The proof is also elementary.

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