ON THE UNIVALENT FUNCTIONS STARLIKE WITH RESPECT TO A BOUNDARY POINT

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ABSTRACT. For the examined functions, we have obtained a structure formula and estimates for \(|f(z)/(1-z)|\) and \(|\arg(f(z)/(1-z))|\), the moduli of the partial sums of the coefficient series and the moduli of the coefficients.

Recently, Robertson [1] introduced the following two classes of univalent functions:

**DEFINITION 1.** Let \(G^*\) denote the class of functions \(f(z)\) analytic in \(D = \{z|z| < 1\}\), normalized so that \(f(0) = 1\), \(f(1) = \lim_{r \to 1} f(r) = 0\), and such that for some real \(\alpha\), \(\text{Re}[e^{i\alpha}f(z)] > 0\), \(z \in D\). In addition let \(f(z)\) map \(D\) univalently on a domain starlike with respect to \(f(1)\). Let the constant function 1 also belong to the class \(G^*\).

**DEFINITION 2.** Let \(G\) denote the class of functions \(f(z)\) analytic and nonvanishing in \(D\), normalized so that \(f(0) = 1\) and such that

\[
\text{Re} \left\{ \frac{2zf'(z)}{f(z)} + \frac{1+z}{1-z} \right\} > 0 \quad (z \in D).
\]


Now we shall continue the study of the class \(G\).

**THEOREM 1.** The function \(f(z)\) belongs to the class \(G\) if and only if \(f(z)\) can be written in the form

\[
f(z) = (1-z) \exp \left\{ - \int_{-\pi}^{\pi} \ln(1-z e^{-it}) d\mu(t) \right\} \quad (z \in D),
\]

for some probability measure \(\mu\) defined on the interval \([-\pi, \pi]\).

**PROOF.** It follows from (1) and a classic result due to Herglotz that

\[
\frac{2zf'(z)}{f(z)} + \frac{1+z}{1-z} = \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-z e^{-it}} d\mu(t)
\]

holds in \(D\) for some probability measure \(\mu(t)\). A simple integration now yields the desired structure formula (2) for \(f(z)\).

**THEOREM 2.** For a fixed \(z \in D\), we have the relation

\[
\{w|w = (1-z)/f(z), f(z) \in G\} = \{w|w - 1 \leq |z|\},
\]

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602
where for \( z \neq 0 \) the equality holds only for the functions

\[
(4) \quad f(z) = \frac{1 - z}{1 - e^{i\omega}z} = 1 + \sum_{n=1}^{\infty} [e^{i\omega} - e^{i(n-1)\omega}]z^n \in G, \quad \omega \in [-\pi, \pi].
\]

**PROOF.** From (2) it follows that

\[
(5) \quad \ln \frac{1 - z}{f(z)} = \int_{-\pi}^{\pi} \ln(1 - ze^{-it})d\mu(t)
\]

holds in \( D \). According to the Carathéodory principle [3] (see also [4, p. 543; 5]), from (5) it follows that for a fixed \( z \in D \) we shall have the relation

\[
(6) \quad \{ \zeta|\zeta = \ln \frac{1 - z}{f(z)}, f(z) \in G \} = \text{CH}\{\zeta|\zeta = \ln(1 - ze^{-it}), t \in [-\pi, \pi]\},
\]

where \( \text{CH} \) denotes the convex hull of the set in the braces. The function

\[
(7) \quad \zeta = \ln w
\]

maps the \( w \)-plane out along the negative real axis onto the strip \( \{ \zeta|-\pi < \text{Im} \zeta < \pi \} \). It is clear geometrically that the function (7) maps the disc \( \{ w| |w-1| \leq |z|, |z| < 1 \} \) onto some convex domain lying in this strip. Therefore, for a fixed \( z \in D \) from (6) the relation

\[
(8) \quad \{ w|w = \frac{1 - z}{f(z)}, f(z) \in G \} = \text{CH}\{w|w = 1 - ze^{-it}, t \in [-\pi, \pi]\}
\]

follows. Now the relation (8) can be written as (3) with (4).

**COROLLARY 2.1.** We have the relation

\[
(9) \quad \bigcup_{f \in G} \{ w|w = \frac{1 - z}{f(z)}, z \in D \} = \{ w| |w - 1| < 1 \}.
\]

**PROOF.** The relation (9) follows from (3) for \( |z| \to 1 \).

**COROLLARY 2.2.** For \( z \in D \) and \( f(z) \in G \), we have the sharp estimates

\[
(10) \quad \frac{1}{1 + |z|} \leq |f(z)/(1 - z)| \leq 1/(1 - |z|)
\]

and

\[
(11) \quad |\arg(f(z)/(1 - z))| \leq \arcsin|z|,
\]

where for \( z \neq 0 \) equality holds only for the functions (4) at the "critical points"

\[
(12) \quad z = \pm|z|e^{-i\omega}
\]

and

\[
(13) \quad z = |z|e^{\pm i(\pi/2 \mp \arcsin|z|)},
\]

respectively.

**PROOF.** The inequalities (10) with (12) and (11) with (13) follow from (3) on the basis of the inequalities \( 1 - |z| \leq |w| \leq 1 + |z| \) and \( |\arg w| \leq \arcsin|z| \), respectively.
COROLLARY 2.3. For each function
\[ f(z) = 1 + d_1 z + d_2 z^2 + \cdots + d_n z^n + \cdots \]
of the class \( G \) in \( D \), the inequalities
\[ |1 + d_1 + d_2 + \cdots + d_n| \leq 1, \quad n = 1, 2, \ldots, \]
and
\[ |d_n| \leq 2, \quad n = 1, 2, \ldots, \]
hold with equality in (15) only for the functions (4) with \( \omega \in [-\pi, \pi] \) and in (16) only for the function (4) with \( \omega = \pm \pi \), i.e., \( f(z) = (1 - z)/(1 + z) \).

PROOF. We write \( w = f(z)/(1 - z) \). Then (9) yields \( \Re w > 1 \), i.e., \( \Re w > 1/2 \). If
\[ w = \sum_{n=0}^{\infty} S_n z^n, \quad S_n = d_0 + d_1 + \cdots + d_n, \quad d_0 = 1, \]
the Borel-Carathéodory inequalities applied to \( 2w - 1 \) yield \( 2|S_n| \leq 2 \), as required.

REMARK. The results in this paper can also be obtained by other methods and results due to Robertson [1, p. 331, Theorem 1; 6, p. 318, Theorem 7; 7, pp. 385–386], Schild [8] and Pinchuk [9, pp. 722, 727–728, 732].

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