THE MACKEY CONTINUITY
OF THE MONOTONE REARRANGEMENT

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ABSTRACT. Let $(A, \mathcal{A}, \mu)$ be a probability space, and let $\text{mes}$ denote the Lebesgue measure on the Borel $\sigma$-algebra $\mathcal{B}$ in $[0,1]$. The nondecreasing-rearrangement operator from the space $L^\infty(\mu) = L^\infty(A, \mathcal{A}, \mu)$ of real-valued essentially bounded functions into $L^\infty = L^\infty([0,1], \mathcal{B}, \text{mes})$ is shown to be uniformly continuous in the Mackey topologies $\tau(L^\infty(\mu), L^1(\mu))$ and $\tau(L^\infty, L^1)$ on $L^\infty(\mu)$ and $L^\infty$, respectively.

1. Definitions and notation. Let $(A, \mathcal{A}, \mu)$ be a probability space and let $\text{mes}$ denote the Lebesgue measure on the $\sigma$-algebra $\mathcal{B}$ of Borel subsets of $[0,1]$. For any real-valued $\mathcal{A}$-measurable function $f$ on $A$ there is a unique, right-continuous, nondecreasing function $f_\uparrow$ on $[0,1]$ such that $\text{mes}(f_\uparrow)^{-1} = \mu f^{-1}$, i.e., $f_\uparrow$ and $f$ are equidistributed. The function $f_\uparrow$ is called the nondecreasing rearrangement of $f$. The nonincreasing rearrangement $f_\downarrow$ of $f$ is defined as $f_\downarrow = -(-f)_\uparrow$, cf. [3, p. 932].

For any $1 \leq p \leq +\infty$, if $f \in L^p(\mu) = L^p(A, \mathcal{A}, \mu)$, then $f_\uparrow \in L^p = L^p([0,1], \mathcal{B}, \text{mes})$. The term "rearrangement operator" is always used to mean the nondecreasing-rearrangement operator $\uparrow$.

The Hardy-Littlewood-Pólya spectral order $<$ is defined as follows: $f < g$ if and only if $\int_0^t f_\uparrow d\text{mes} \leq \int_0^t g_\downarrow d\text{mes}$ for all $t \in [0,1]$ with equality for $t = 0$.

For a pair of vector spaces $E$ and $F$ in duality, let $\sigma(E,F)$ and $\tau(E,F)$ denote, respectively, the weak and Mackey topologies on $E$ induced by $F$. The duality between $L^p(\mu)$ and $L^p'(\mu)$, where $p^{-1} + (p')^{-1} = 1$, is $\langle f, g \rangle_\mu = \int_A fg \, d\mu$, and the subscript $\mu$ is suppressed when $\mu = \text{mes}$.

2. The Mackey continuity of the rearrangement operator. For $1 \leq p < +\infty$, the rearrangement operator $\uparrow$ is a nonexpansive mapping of $L^p(\mu)$ into $L^p$ (with their usual norms). This result is contained in [3, Corollary (5.2)] with $\phi(x, y) = -|x - y|^p$; note also [3, Remark (5.7)]. Alternatively, it can be seen from the relation $f_\uparrow \downarrow - g_\downarrow < f - g$ [3, (6.1)(ii)], which, in view of the monotonicity of the $L^p$-norm in the preorder $<$ [2, Theorem (2.5)], implies $\|f_\uparrow - g_\downarrow\|_p \leq \|f - g\|_p$. Note also that for $p = 2$ the nonexpansive property of the operator $\uparrow$ follows directly from the Hardy-Littlewood-Pólya inequality $\langle f, g \rangle_\mu \leq \langle f_\uparrow, g_\downarrow \rangle$.

For $p = +\infty$, the rearrangement operator is nonexpansive in the essential supremum norm. For an economic application in [6], however, it is necessary to establish the continuity of the rearrangement operator in some topologies on $L^\infty(\mu)$ and $L^\infty$ in which the respective continuous duals are $L^1(\mu)$ and $L^1$. From the following example it is seen that the rearrangement operator on $L^\infty$ into $L^\infty$ is discontinuous in the weak topology $\sigma(L^\infty, L^1)$. 

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Example. Let $f^n(t) = \text{sgn} \sin 2\pi nt$ for $t \in [0,1]$ and $n = 1, 2, \ldots$. One has $f^n \to 0$ in $\sigma(L^\infty, L^1)$; but for all $n$, $f^n$ is equal to $(-1)$ on $[0,1/2)$ and to $+1$ on $[1/2, 1]$.

As this example suggests, continuity holds for the Mackey topology.

Theorem. The rearrangement operator $\uparrow$ is a uniformly continuous mapping of $L^\infty(\mu)$ endowed with the Mackey topology $\tau(L^\infty(\mu), L^1(\mu))$ into $L^\infty$ endowed with the Mackey topology $\tau(L^\infty, L^1)$.

Proof. The Mackey topology $\tau(E, F)$ is the topology of uniform convergence on $\sigma(F, E)$-compact sets (see, for example, [9, Corollary 1 to Theorem IV.3.2, and subsequent remarks]). By the Dunford-Pettis compactness criterion, weak relative compactness is equivalent to uniform integrability (see [5, Theorems IV.8.9 and V.6.1]).

Take any uniformly integrable $\Xi \subset L^1$. To prove the Theorem, it is necessary and sufficient to find a uniformly integrable $\Psi \subset L^1(\mu)$ such that, for any $f, g \in L^\infty(\mu)$, if $|\langle \psi, g - f \rangle_\mu| < 1$ for all $\psi \in \Psi$, then $|\langle \phi, g - f \rangle| < 1$ for all $\phi \in \Phi$.

From [3, (6.1)(ii)],

\begin{equation}
 g_\uparrow - f_\uparrow < g - f \quad \text{and} \quad f_\uparrow - g_\uparrow < f - g.
\end{equation}

By the Hardy-Littlewood-Pólya inequality (see, for example, [4, Lemma (3.4)]),

\begin{equation}
 \langle \phi, g_\uparrow - f_\uparrow \rangle \leq \langle \phi_\uparrow, (g - f)_\uparrow \rangle \quad \text{for all } \phi \in L^1
\end{equation}

and

\begin{equation}
 \langle \phi, f_\uparrow - g_\uparrow \rangle \leq \langle \phi_\uparrow, (f - g)_\uparrow \rangle \quad \text{for all } \phi \in L^1.
\end{equation}

This last inequality can also be written equivalently as

\begin{equation}
 \langle \phi, g_\uparrow - f_\uparrow \rangle \geq \langle \phi_\uparrow, (g - f)_\uparrow \rangle \quad \text{for all } \phi \in L^1.
\end{equation}

Consider first the case of a nonatomic $\mu$. Let $S$ be the set of all measure-preserving mappings of $(A, \mathcal{A}, \mu)$ into $[0,1]$. By [4, Proposition (3.3) or 8, Lemma 1], for any $f$ and $g$ there exists $S \in S$ such that $g - f = (g - f)_\uparrow oS$. Then for such $S$,

\begin{equation}
 \langle h, (g - f)_\uparrow \rangle = \langle h o S, g - f \rangle_\mu \quad \text{for all } h \in L^1.
\end{equation}

Using (2.6) for $h = \phi_\uparrow, \phi_\uparrow$ with (2.4) and (2.5) gives

\begin{equation}
 \inf_{S \in S} \langle \phi_\uparrow o S, g - f \rangle_\mu \leq \langle \phi_\uparrow, g_\uparrow - f_\uparrow \rangle \leq \sup_{S \in S} \langle \phi_\uparrow o S, g - f \rangle_\mu \quad \text{for all } \phi \in L^1.
\end{equation}

Consider the set $\Phi_\uparrow o S = \{ \phi_\uparrow o S | \phi \in \Phi, S \in S \}$: It is a uniformly integrable subset of $L^1(\mu)$ because uniform integrability is a property which depends only on the distributions of the functions involved. It follows from (2.7) that it suffices to take $\Psi = \Phi_\uparrow o S$. This completes the proof for a nonatomic $\mu$.

Only a slight modification of the preceding argument has to be made for a $\mu$ with atoms. By [7, Theorem (7.1) or 3, p. 390, first line], there exist $\bar{\phi}, \bar{\phi} \in L^1(\mu)$ such that

\begin{equation}
 \langle \phi_\uparrow, (g - f)_\uparrow \rangle = \langle \bar{\phi}, g - f \rangle_\mu \quad \text{and} \quad \bar{\phi} < \phi,
\end{equation}

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and

\[ \langle \phi_1, (g - f)_\uparrow \rangle = \langle \phi, g - f \rangle_\mu \quad \text{and} \quad \phi < \phi. \]

Let \( \Omega(\phi) = \{ \psi \in L^1(\mu) | \psi < \phi \} \) and \( \Omega(\Phi) = \bigcup_{\phi \in \Phi} \Omega(\phi) \). By [1, Corollary 4.3 and [7, Theorem 9.5, the set \( \Omega(\Phi) \) is uniformly integrable because \( \Phi \) is. From (2.4), (2.5), (2.8) and (2.9) it follows that

\[ \inf_{\psi \in \Omega(\Phi)} \langle \psi, (g - f)_\mu \rangle \leq \langle \phi, (g - f)_\mu \rangle \]

\[ \leq \sup_{\psi \in \Omega(\Phi)} \langle \psi, (g - f)_\mu \rangle \quad \text{for all} \ \phi \in \Phi. \]

It follows from (2.10) that it suffices to take \( \Psi = \Omega(\Phi) \). Q.E.D.

(2.11) REMARK. For countable sequences, convergence in the Mackey topology \( \tau(L^\infty(\mu), L^1(\mu)) \) is equivalent to convergence in the measure \( \mu \) and boundedness in the essential supremum norm. Therefore, sequential continuity of the rearrangement operator in the Mackey topologies under consideration follows also from its continuity in the topologies of convergence in the measures \( \mu \) and mes.

(2.12) REMARK. Even after their restriction to the range of the operator \( \uparrow \), viz., the cone of nondecreasing functions in \( L^\infty \), the Mackey topology \( \tau(L^\infty, L^1) \) is strictly stronger than the weak topology \( \sigma(L^\infty, L^1) \).

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