PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 3, July 1986

A NOTE ON UNIFORM OPERATORS

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ABSTRACT. An operator is uniform if its restriction to any infinite-dimensional invariant subspace is unitarily equivalent to itself. We show that a uniform operator having a proper infinite-dimensional invariant subspace resembles an analytic Toeplitz operator in the way that the weakly closed algebra generated by it and the identity operator is isomorphic to a subalgebra of the Calkin algebra; furthermore, this algebra contains no nonscalar operator which is quasi-similar to a normal operator.

1. Introduction. A bounded linear operator T on a separable complex Hilbert space is said to be uniform if for every infinite-dimensional invariant subspace M of T, the restriction $T|M$ is unitarily equivalent to T. Since scalar operators are trivial examples of uniform operators, we restrict our attention to nonscalar operators. Thus in this paper, all uniform operators are assumed to be nonscalar operators; and the term "invariant subspace" means proper invariant subspace for convenience. The following theorem is proved in Wang and Stampfli [**5**].

THEOREM. *Let T be a uniform operator. If T has an infinite-dimensional invariant subspace, then T has no finite-dimensional invariant subspace.*

According to this theorem, there are two mutually disjoint classes of uniform operators. The first class contains those uniform operators which have infinite-dimensional invariant subspaces but no eigenvalue. The second class contains uniform operators with no infinite-dimensional invariant subspace. The unilaternal shift is an example of the first class, while the Donoghue operators are in the second.

Cowen and Douglas conjecture in [**1**] that every operator in the first class is unitarily equivalent to an analytic Toeplitz operator. So far, we have proved in [**5**] that the conjecture is true for a uniform operator with nonempty compression spectrum; but the general problem is left open. However, in this note, we show that a uniform operator T in the first class resembles an analytic Toeplitz operator in the way that the weakly closed algebra generated by the identity operator and T has the following properties: (1) It contains no nonscalar operator quasi-similar to a normal operator. (2) It contains no nonzero compact operator, so that it is isomorphic to a subalgebra of the Calkin algebra.

If H is a Hilbert space, then $B(H)$ denotes the algebra of all bounded linear operators. If T belongs to $B(H)$, $A(T)$ denotes the weakly closed algebra generated

Received by the editors March 27, 1984 and, in revised form, July 26, 1985.
1980 *Mathematics Subject Classification*. Primary 47B35, 47B37.
Key words and phrases. Uniform operator, Calkin algebra, invariant subspace, normal operator, quasi-similarity, Toeplitz operator.

[1] This paper is part of the author's dissertation written under the direction of Professor J. G. Stampfli at Indiana University.

by the identity operator and T. If M is an invariant subspace of T, $T|M$ denotes the restriction of T to M. All Hilbert spaces are assumed to be separable complex.

2. Quasi-similarity. An operator X from a Hilbert space H into a Hilbert space K is said to be quasi-invertible if X has zero kernel and dense range. An operator A on H is said to be a quasi-affine transform of an operator B on K if there is a quasi-invertible operator X from H into K such that $BX = XA$. A and B are quasi-similar if they are quasi-affine transforms of one another. For our result on the quasi-similarity with normal operators, we need a theorem of Douglas [**2**].

THEOREM 2.1. *If a normal operator N_1 is a quasi-affine transform of a normal operator N_2, then N_1 is unitarily equivalent to N_2.*

Let T be an operator on H. A subspace M is said to be a hyperinvariant subspace of T if M is invariant for every operator in the commutant of T. In general, the operators in $A(T)$ may not be uniform when T is uniform. For example, the square of the unilateral shift is not uniform; however, the restriction of this operator to any hyperinvariant subspace is unitarily equivalent to itself.

LEMMA 2.2. *Let T be a uniform operator on H. If $S \in A(T)$ and M is an infinite-dimensional hyperinvariant subspace of S, then $S|M$ is unitarily equivalent to S.*

PROOF. Since T commutes with S, M is an invariant subspace of T. Since M is infinite-dimensional, $T|M$ is unitarily equivalent to T. Thus we can find a unitary map U from M onto H such that $TM = U^{-1}TU$. It follows that $P(T)|M = U^{-1}P(T)U$ for any polynomial P. Since $S \in A(T)$, we can find a net of polynomials $\{P_\alpha : \alpha \in \Lambda\}$ such that $\{P_\alpha(T) : \alpha \in \Lambda\}$ converges to S weakly. It follows that $\{P_\alpha(T)|M : \alpha \in \Lambda\} = \{U^{-1}P_\alpha(T)U : \alpha \in \Lambda\}$ converges to $U^{-1}SU$ weakly. Hence $S|M = U^{-1}SU$. Q.E.D.

Hoover proves in [**3**] that if an operator A is quasi-similar to an operator B and A has a hyperinvariant subspace, then B also has one. In the proof of the next theorem, we use his method of producing hyperinvariant subspaces.

THEOREM 2.3. *Let T be a uniform operator on H. If $S \in A(T)$ and S is quasi-similar to a normal operator, then S is a scalar operator.*

PROOF. Let S be a nonscalar operator in $A(T)$ where S is quasi-similar to a normal operator N on H. Then there exist two quasi-invertible operators X and Y on H such that $SX = XN$ and $YS = NY$. Since S is nonscalar, so is N. Let M be a proper spectral subspace of N, then M is hyperinvariant for N by the spectral theorem. Thus if M' is the closed linear span of $\{CXx : x \in M$ and C is any operator in the commutant of $S\}$, then M' is hyperinvariant for S and $YM' \subseteq M$; see Hoover [**3**].

Note that since $ST = TS$, M' is an invariant subspace of T. We have two cases.

Case 1. T has no infinite-dimensional invariant subspace. In this case, M' is finite-dimensional. Since X is injective and $XM \subseteq M'$, M is finite-dimensional too. Therefore every proper spectral subspace of N is finite-dimensional. Since for a spectral subspace, M, $H \ominus M$ is also a spectral subspace, we have a contradiction.

Case 2. T has an infinite-dimensional invariant subspace. By the theorem in the introduction, M' is infinite-dimensional. Since Y is injective and $YM' \subseteq M$, M

is infinite-dimensional too. Furthermore, since YX commutes with N, M reduces YX by the spectral theorem. It follows that $YX|M$ is quasi-invertible because YX is. Thus we obtain the needed fact that $Y|M'$ is quasi-invertible. Now we have

$$(Y|M')(S|M') = (N|M)(Y|M').$$

Thus $S|M'$ is a quasi-affine transform of $N|M$. By Lemma 2.2, $S|M'$ is unitarily equivalent to S. Furthermore, S is known to be quasi-similar to N. It follows transitively that N is a quasi-affine transform of $N|M$. Douglas' Theorem 2.1 applies here to give us the fact that N is unitarily equivalent to $N|M$ for every proper spectral subspace M. It follows that the spectrum of N is a singleton and N is a scalar operator, a contradiction. Q.E.D.

It should be mentioned that the preceding theorem is true for both classes of uniform operators. The result in the next section holds for the first class only.

3. Embedding into the Calkin algebra. It is well known that every compact operator has a hyperinvariant subspace; see Lomonsov [**4**]. Let T be a uniform operator with an infinite-dimensional invariant subspace and S a nonzero compact operator in $A(T)$. First of all, Lomonosov's result produces a hyperinvariant subspace for S. Starting with this space, the uniformness of T produces a lot more for S, in fact, a decreasing sequence of infinite-dimensional invariant subspaces of S; furthermore, S behaves the same on each of them. This will contradict the compactness if S is not zero. Thus we have the following theorem.

THEOREM 3.1. *If T is a uniform operator with an infinite-dimensional invariant subspace, then the canonical map from $A(T)$ into the Calkin algebra is an embedding.*

PROOF. It suffices to show that there is no nonzero compact operator in $A(T)$. Assume the contrary—that S is a nonzero compact operator in $A(T)$. Then by Lomonosov's theorem, S has a proper hyperinvariant subspace M_1. Since $ST = TS$, M_1 is a proper invariant subspace of T. Therefore, M_1 is infinite-dimensional because T allows no proper finite-dimensional invariant subspace.

By Lemma 2.2, $S|M_1$ is unitarily equivalent to S. Thus there is a unitary map U from H onto M_1 such that $(S|M_1)U = US$. Let e_0 be a unit vector in H which is orthogonal to M_1. For $n = 1, 2, \ldots$, set $M_n = U^n M_1$ and $e_n = U^n e_0$. Since e_0 is orthogonal to M_1, it is easy to see that for each n, e_n is orthogonal to M_{n+1}.

Thus we have a sequence of invariant subspaces of S, $H = M_0 \subseteq M_1 \subseteq M_2, \ldots$ and an orthonormal sequence $\{e_n | n = 0, 1, 2, \ldots\}$ such that

$$(S|M_n)U^n = U^n S, \quad U^n e_0 = e_n \quad \text{for } n = 0, 1, 2, \ldots.$$

It follows that $Se_n = (S|M_n)U^n e_0 = U^n S e_0$. Hence $\|Se_n\| = \|Se_0\|$ for $n = 0, 1, 2, \ldots$. At the same time, since S is compact, $\{Se_n\}$ converges to 0 strongly; that is, $\{\|Se_n\|\}$ converges to 0, a contradiction. Therefore, $A(T)$ contains no nonzero compact operator. Then the canonical map from $A(T)$ into the Calkin algebra is an embedding. Q.E.D.

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