COMPACT DERIVATIONS OF NEST ALGEBRAS

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Abstract. In this paper we determine all the weakly compact derivations of a nest algebra. We also obtain necessary and sufficient conditions in order that a nest algebra admit compact derivations. Finally we prove that every compact derivation of a nest algebra \( \mathcal{A} \) is the norm limit of finite-rank derivations.

1. Let \( \mathcal{A} \) be a Banach algebra and let \( X \) be a Banach \( \mathcal{A} \)-module. By an \( X \)-valued derivation of \( \mathcal{A} \) we mean a linear mapping \( \delta: \mathcal{A} \to X \) with the property \( \delta(ab) = a \delta(b) + \delta(a)b \) for all \( a \in \mathcal{A} \), \( b \in \mathcal{A} \). The derivation \( \delta \) is called compact if \( \delta \) is a compact operator between the Banach space \( \mathcal{A} \) and \( X \), and weakly compact if \( \delta \) is a weakly compact operator from \( \mathcal{A} \) to \( X \) (i.e. \( \delta(\mathcal{A}_1) \) is relatively weakly compact in \( X \), where \( \mathcal{A}_1 \) is the unit ball of \( \mathcal{A} \) [4]). Let \( H \) be a complex Hilbert space, \( B(H) \) the algebra of all bounded operators on \( H \) and \( \mathcal{K}(H) = \mathcal{K} \) the set of compact operators on \( H \). In [7] Johnson and Parrott investigated derivations of a von Neumann subalgebra of \( B(H) \) with range contained in \( \mathcal{K} \). They proved that in most cases such derivations are implemented by a compact operator. The general result was recently obtained by Popa [8] who proved that this is the case for all von Neumann subalgebras of \( B(H) \). Such derivations are known to be weakly compact [1].

On the other hand, in a series of papers [1, 9, 10], C. A. Akemann, S. K. Tsui, and S. Wright have determined the structure of all compact and weakly compact \( \mathcal{A} \)-valued derivations of a C*-algebra \( \mathcal{A} \), and of all compact \( B(H) \) valued derivations of a C*-subalgebra of \( B(H) \).

In this note we determine the structure of all \( \mathcal{A} \)-valued compact and weakly compact derivations of a nest algebra \( \mathcal{A} \). In particular we prove that every compact derivation of a nest algebra \( \mathcal{A} \) is the norm limit of finite rank derivations. We need the following result.

Lemma 1 [6]. Let \( H \) be an infinite dimensional Hilbert space. If \( \delta \) is a compact derivation of \( B(H) \) then \( \delta \equiv 0 \).
2. In this section we state and prove our results on nest algebras.

Let $\mathcal{N} \cap \mathcal{B}(H)$ be a complete nest of projections, i.e., a totally ordered set of orthogonal projections which is strongly closed. We denote by $\mathcal{N}$ the nest algebra

$$\text{alg.}\mathcal{N} = \{ a \in \mathcal{B}(H) | \text{pap} = ap, p \in \mathcal{N} \}.$$ 

If $\mathcal{N} \cap B(H)$ is the set of compact operators on $H$ let $\mathcal{K}(\mathcal{N}) = \mathcal{N} \cap \mathcal{K}$. It is known [5] that $\mathcal{K}(\mathcal{N})$ is ultraweakly dense in $\mathcal{N}$. A simple consequence of this and the duality between compact operators, trace class operators and $B(H)$ is the following.

**Lemma 2.** $\mathcal{K}(\mathcal{N})^{**} = \mathcal{N}$ (where $\mathcal{K}(\mathcal{N})^{**}$ denotes the bidual of $\mathcal{K}(\mathcal{N})$).

We state now the analogue of [1, Theorem 3.1] for derivations of nest algebras.

**Theorem 3.** Let $\delta$ be a derivation of $\mathcal{N}$. The following conditions are equivalent:

1. $\delta$ is weakly compact.
2. The range of $\delta$ is contained in $\mathcal{K}(\mathcal{N})$.
3. $\delta = \text{ad}(k)$ with $k \in \mathcal{K}(\mathcal{N})$.

**Proof.** (1) $\Rightarrow$ (2) By [2] $\delta$ is inner, and then $\delta(\mathcal{K}(\mathcal{N})) \subseteq \mathcal{K}(\mathcal{N})$. By Lemma 2, $\mathcal{K}(\mathcal{N})^{**} = \mathcal{N}$ whence $\delta = (\delta|_{\mathcal{K}(\mathcal{N})})^{**}$. Since $\delta$ is weakly compact, $\delta|_{\mathcal{K}(\mathcal{N})}$ is too, so by [4, Theorem VI.4.2] it follows that $\delta(\mathcal{N}) \subseteq \mathcal{K}(\mathcal{N})$.

(2) $\Rightarrow$ (3) follows immediately from [3, Theorem 1.4 and 2].

(3) $\Rightarrow$ (1) follows from the same implication of [1, Theorem 3.3].

The preceding result gives necessary and sufficient conditions for a derivation of a nest algebra to be weakly compact. Further, (Theorem 8) we give necessary and sufficient conditions for a nest algebra to admit a nonzero compact derivation.

We need the following well-known (and easy to prove) result.

**Lemma 4.** Let $\mathcal{N}$ be a nest algebra and let $\mathcal{J} = \text{lat} \mathcal{N}$. If $p \in \mathcal{N}$ then

$$pB(X)(1 - p) \subseteq \mathcal{N}.$$ 

**Lemma 5.** If $\bigwedge\{ p \mid p \in \mathcal{N}, p \neq 0 \} = 0$ then $\mathcal{N}$ does not admit nonzero compact derivations.

**Proof.** The condition of the lemma implies $\dim pH = \infty$ for all $p \in \mathcal{N}, p \neq 0$. Let $\delta$ be a compact derivation of $\delta$. By Theorem 3, $\delta = \text{ad}(k)$ for some $k \in \mathcal{K}(\mathcal{N})$.

We consider the following two possibilities:

(a) There is $p_0 \in \mathcal{N}$, and $\xi_0 \in (1 - p_0)H$, $||\xi_0|| = 1$ such that $\langle k\xi_0, \xi_0 \rangle \neq 0$ (here $\langle \ldots, \ldots \rangle$ denotes the inner product in $H$). Let $k\xi_0 = \lambda \xi_0 + \eta_0$, where $\langle \xi_0, \eta_0 \rangle = 0$ and $\lambda \neq 0$. Since $\dim p_0H = \infty$, let $\{ \xi_n \}_{n=1}^{\infty}$ be an orthonormal family of that space. We consider the operators $u_n: (1 - p_0)H \to p_0H$ defined by

$$u_n\xi_0 = \xi_n, \quad u_n[\xi_0] = 0, \quad n = 1, 2, \ldots.$$ 

By Lemma 4, $u_n \in \mathcal{N}, n = 1, 2, \ldots$. Since $||u_n|| = 1$ for all $n$ and $\delta$ is compact, the sequence $\{ \delta(u_n) \}$ contains a (norm) convergent subsequence which will be denoted by $\{ \delta(u_n) \}$ too. On the other hand,

$$\delta(u_n)\xi_0 = ku_n\xi_0 - u_nk\xi_0 = k\xi_0 - \lambda\xi_n.$$
Since $k$ is compact and $\lambda \neq 0$, this sequence does not contain any convergent subsequence. This contradiction shows that in this case $k$ and so $\delta$ must be equal to zero.

(b) $\langle k\xi, \xi \rangle = 0$ for all $\xi \in (1-p)H$ and all $p \in \mathcal{N}$, $p \neq 0$. In this case $(1-p)k = 0$ for all $p \in \mathcal{N}$ and since $k$ is compact, it follows that $k = 0$ so $\delta = 0$.

We denote $\mathcal{A}^* = \{ a^* \in B(H) | a \in \mathcal{A} \}$. Then $\mathcal{A}^*$ is a nest algebra, and $\text{Lat} \mathcal{A}^* = \{ 1 - p | p \in \mathcal{N} \}$.

**Lemma 6.** $\text{ad}(k)$, $k \in \mathcal{K}(\mathcal{A})$, is a compact derivation of $\mathcal{A}$ if and only if $\text{ad}(k^*)$ is a compact derivation of $\mathcal{A}^*$.

**Proof.** The proof is obvious.

**Lemma 7.** If $\Lambda \{ 1 - p | p \in \mathcal{N}, p \neq 1 \} = 0$ then $\mathcal{A}$ does not admit nonzero compact derivations.

**Proof.** The proof follows from Lemma 5 applied to $\mathcal{A}^*$, and Lemma 6.

**Lemma 8.** Let $\delta = \text{ad}(k)$, $k \in \mathcal{K}(\mathcal{A})$, be a compact derivation of a nest algebra $\mathcal{A}$. Then we have

(a) If $q_0 \in \mathcal{N}$ is such that $\dim (1 - q_0)H = \infty$ then $kq_0 = 0$.

(b) If $p_0 \in \mathcal{N}$ is such that $\dim p_0H = \infty$, then $(1 - p_0)k = 0$.

**Proof.** (1) Suppose $kq_0 \neq 0$. Then there is $\xi_0 \in q_0H$ with $\|\xi_0\| = 1$, $\xi_0 \neq 0$, such that $k\xi_0 = \xi_0$. Since $\dim (1 - q_0)H = \infty$, there is an infinite orthonormal family $\{ \xi_n \}_{n=1}^{\infty}$ of that space. Consider the following family $\{ u_n \}$ of operators, $u_n$: $(1 - q_0)H \rightarrow q_0H$:

$$u_n \xi_n = \xi_0, \quad u_n [\xi_n]^1 = 0, \quad n = 1, 2, \ldots.$$  

Obviously, $\|u_n\| = 1$, $n = 1, 2, \ldots$. By Lemma 4 $u_n \in \mathcal{A}$ for all $n$. Since $\text{ad}(k)$ is compact, the sequence $\{ ku_n - u_n k \}_{n=1}^{\infty}$ contains a (norm) convergent subsequence which will be denoted by $\{ ku_n - u_n k \}_{n=1}^{\infty}$ too. Therefore

$$\lim_{m \gg n \rightarrow \infty} \|k(u_n - u_m) - (u_n - u_m)k\| = 0.$$  

In particular

$$\lim_{m \gg n \rightarrow \infty} \|k(u_n - u_m)\xi_n - (u_n - u_m)k\xi_n\| = 0.$$  

Since $k$ is compact and $\{ \xi_n \}$ is an orthonormal family, we have $\lim_{n \rightarrow \infty} \|k\xi_n\| = 0$. Since $\|u_n\| = 1$ for all $n$ we also have $\lim_{m \gg n \rightarrow \infty} \|(u_n - u_m)k\xi_n\| = 0$, so

$$\lim_{m \gg n \rightarrow \infty} \|k(u_n - u_m)\xi_n\| = 0.$$  

But $k(u_n - u_m)\xi_n = ku_n\xi_n = k\xi_0 = \xi_0 \neq 0$. This contradiction proves the lemma.

To prove (2) we apply the preceding argument to $\mathcal{A}^*$ and $1 - p_0$ (instead of $\mathcal{A}$ and $q_0$). We now state our necessary and sufficient conditions for $\mathcal{A}$ to admit a nonzero compact derivation.
Theorem 9. Let $\mathcal{A}$ be a nest algebra and $\mathcal{N} = \text{Lat} \mathcal{A}$. The following are equivalent:

(1) $\mathcal{A}$ admits a nonzero compact derivation.

(2) There is a $p \in \mathcal{N}$, $p \neq 0$, with $\dim pH < \infty$ and a $q \in \mathcal{N}$, $q \neq 1$, such that $\dim(1 - q)H < \infty$.

Proof. (2) $\Rightarrow$ (1) Indeed let $k \in pB(H)(1 - q) \subset \mathcal{A}$. Then, it is easy to see that $\text{ad}(k)$ is a compact (actually a finite rank) derivation of $\mathcal{A}$.

(1) $\Rightarrow$ (2) Suppose first that $\dim pH = \infty$ for all $p \in \mathcal{N}$, $p \neq 0$. Let $p_0 = \bigwedge \{ p \in \mathcal{N} \mid p \neq 0 \}$. Then, either $p_0 = 0$ or $\dim p_0H = \infty$. If $p_0 = 0$ then by Lemma 5, $\mathcal{A}$ does not admit nonzero compact derivations. Assume $p_0 \neq 0$ and let $\delta$ be a compact derivation of $\mathcal{A}$. By Theorem 3, $\delta = \text{ad}(k)$ for some $k \in \mathcal{N}(\mathcal{A})$. Since $\mathcal{A}$ is a nest algebra, it follows that $p_0 \mathcal{A} p_0 = B(p_0H)$. Since $kp_0$ implements a compact derivation of $p_0 \mathcal{A} p_0 = B(p_0H)$ and $\dim p_0H = \infty$, by Lemma 1 we have $kp_0 = p_0 kp_0 = 0$. On the other hand, by Lemma 8, we have $(1 - p_0)k = 0$. Therefore $k = p_0 k (1 - p_0)$. It remains to prove that $p_0 k (1 - p_0) = 0$. Assume the contrary. Then there is $\xi \in (1 - p_0)H$, $\xi \neq 0$, with $k\xi = \xi_0 \in p_0H$, $\|\xi_0\| = 1$. Since $\dim p_0H = \infty$, let $\{\xi_n\}_{n=1}^{\infty}$ be an orthonormal family of that infinite dimensional space. Let $\{u_n\}_{n=1}^{\infty} \subset B(p_0H) = p_0 \mathcal{A} p_0 \subset \mathcal{A}$, $\|u_n\| = 1$ be the family of operators defined by

$$u_n \xi_0 = \xi_n, \quad u_n [\xi_0] = 0, \quad n = 1, 2, \ldots.$$ 

Then, since $\text{ad}(k)$ is a compact derivation of $\mathcal{A}$, the sequence $\{(ku_n - u_n k)\xi\}_{n=1}^{\infty}$ contains a norm convergent subsequence. But $(ku_n - u_n k)\xi = -u_n k\xi = -\xi_n$ so this sequence does not contain any convergent subsequence. This contradiction shows that $p_0 k (1 - p_0) = 0$ and hence $k = 0$. Assume now that $\dim(1 - q)H = \infty$ for all $q \in \mathcal{N}$, $q \neq 1$. Then, the preceding argument applied to $\mathcal{A}^*$ shows that $\mathcal{A}^*$ does not admit nonzero compact derivations. By Lemma 6, $\mathcal{A}$ does not admit nonzero compact derivations and the proof of Theorem 9 is complete.

Theorem 10. Every compact derivation of a nest algebra $\mathcal{A}$ is the norm limit of finite rank derivations.

Proof. Let

$$p_0 = \bigvee \{ p \in \mathcal{N} \mid \dim pH < \infty \} \quad \text{and} \quad q_0 = \bigwedge \{ q \in \mathcal{N} \mid \dim(1 - q)H < \infty \}.$$ 

Let also $\delta = \text{ad}(k)$, $k \in \mathcal{N}(\mathcal{A})$, be a compact derivation of $\mathcal{A}$. We first prove that $k = p_0 k (1 - q_0)$. We note that $(1 - p_0)k$ implements a compact derivation of the nest algebra $(1 - p_0) \mathcal{A} (1 - p_0)$. If $\dim p_0 < \infty$ then $\text{Lat}(1 - p_0) \mathcal{A} (1 - p_0)$ does not contain any finite dimensional projection, so by Theorem 9, $(1 - p_0)k = 0$. If $\dim p_0 = \infty$ we have $(1 - p_0)k = 0$ by Lemma 8(b). Hence $k = p_0 k$. A similar argument shows that $k = k(1 - q_0)$. From the definition of $p_0$ (respectively $q_0$) it follows that there exists a sequence $\{p_n\} \subset \mathcal{N}$, $\dim p_n H < \infty$ (respectively $\{q_n\} \subset \mathcal{N}$, $\dim(1 - q_n)H < \infty$) such that $p_n \uparrow p_0$ (respectively $q_n \downarrow q_0$); of course, if $\dim p_0 H < \infty$ (respectively $\dim(1 - q_0)H < \infty$), the sequence $\{p_n\}$ (respectively
\( \{q_n\} \) will be constant. Since \( k \) is compact, it follows that
\[
k = p_0 k (1 - q_0) = (\text{norm}) \lim_{n \to \infty} p_n k (1 - q_n).
\]
As obviously \( \text{ad}(p_n k (1 - q_n)) \) is finite rank the proof of Theorem 10 is complete.

**References**