SPURIOUS BROWNIAN MOTIONS

SÁNDOR CSÖRGŐ

Abstract. Spurious Brownian motions are characterized in $\mathbb{R}^d$, $d \geq 2$.

Let $W = \{W(t) = (W_1(t), \ldots, W_d(t)), t \geq 0\}$ be an $\mathbb{R}^d$-valued mean-zero Gaussian process such that all projections $Y_\lambda(t) = \sum_{j=1}^d \lambda_j W_j(t)$ behave as if $W$ were standard Brownian motion in $\mathbb{R}^d$, i.e., $EY_\lambda(s)Y_\lambda(t) = \min(s,t)\sum_{j=1}^d \lambda_j^2$. Following Hardin [1], we call $W$ a spurious Brownian motion if it is not a standard Brownian motion in $\mathbb{R}^d$. He showed by an example that such a process exists in $\mathbb{R}^2$.

Let $S(s,t) = (\sigma_{jk}(s,t))$ denote the covariance matrix function of $W$: $\sigma_{jk}(s,t) = EW_j(s)W_k(t)$, $j, k = 1, \ldots, d$. Necessarily, $\sigma_{jj}(s,t) = \min(s,t)$ for each $j = 1, \ldots, d$. Hence the identity $EY_\lambda(s)Y_\lambda(t) = \sum_{j,k=1}^d \lambda_j \lambda_k \sigma_{jk}(s,t) = \min(s,t)\sum_{j=1}^d \lambda_j^2$ is equivalent to the identity

$$\sum_{j,k=1, j \neq k}^d \lambda_j \lambda_k \sigma_{jk}(s,t) = 0.$$

This holds for any vector $\lambda = (\lambda_1, \ldots, \lambda_d)$ if and only if $\sigma_{jk}(s,t) = -\sigma_{kj}(s,t)$ for all $j, k = 1, \ldots, d$; $j \neq k$. Thus $W$ is a spurious Brownian motion if and only if $S(s,t)$ is skew-symmetric and not diagonal.

Hardin’s example in $\mathbb{R}^2$ is an example for such a covariance matrix with $\sigma_{12}(s,t) = 3^{-1}(\min(2s,t) - \min(s,2t))$. However, for any choice of the functions $\sigma_{jk}(s,t)$, $k = j + 1, \ldots, d$; $j = 1, \ldots, d$, such that $|\sigma_{jk}(s,t)| \leq (st)^{1/2}$, $\sigma_{j,j}(s,t) + \sigma_{jk}(t,s) = 0$, and that $\sigma_{jk}(s,t) \neq 0$ for at least one pair $(j, k)$, there is a spurious Brownian motion in $\mathbb{R}^d$, provided that for any integer $m \geq 2$, any $t_1, \ldots, t_m \geq 0$, and any real numbers $\lambda_{ij}$, $i = 1, \ldots, d$; $l = 1, \ldots, m$, we have

$$\sum_{i,j=1}^m \left( \min(t_i,t_j) \sum_{j=1}^d \lambda_{ji}\lambda_{jl} + \sum_{j=1}^{d-1} \sum_{k=j+1}^d \sigma_{jk}(t_i,t_j)(\lambda_{ji}\lambda_{kl} - \lambda_{jl}\lambda_{ki}) \right) \geq 0.$$

These conditions are necessary and sufficient for the existence of a mean-zero stochastic process $X = \{X(t) = (X_1(t), \ldots, X_d(t)), t \geq 0\}$ that has covariance matrix function defined as the skew-symmetric matrix corresponding to the functions $\sigma_{jk}(s,t)$, $k = j + 1, \ldots, d$; $j = 1, \ldots, d$, and having diagonal $\sigma_{jj}(s,t) = \min(s,t)$,

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The projections of $X$ already behave as if $X$ were a standard Brownian motion in $R^d$, though $X$ is not necessarily Gaussian. If $X$ is Gaussian, and such an $X$ exists, then it is necessarily a spurious Brownian motion in $R^d$.

**References**


**Bolyai Institute, Szeged University, 6720 Szeged, Hungary**