U-EMBEDDED SUBSETS OF NORMED LINEAR SPACES
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ABSTRACT. A subset $S$ of a metric space $X$ is $U$-embedded in $X$ if every uniformly continuous function $f: S \to R$ extends to a uniformly continuous function $F: X \to R$. Thus $U$-embedding is the uniform analogue of $C$-embedding. The Tietze extension theorem tells us exactly which subsets of metric spaces are $C$-embedded. The uniform analogue would tell us exactly which subsets of metric spaces are $U$-embedded. In this paper, a characterization of $U$-embedded subsets of the Euclidean plane (or any normed linear space) is given.

A subset $S$ of a uniform space $X$ is $U$-embedded in $X$ if every real-valued uniformly continuous function $f: S \to R$ extends to a uniformly continuous function $F: X \to R$. Thus, $U$-embedding is the uniform analogue of $C$-embedding in topological spaces. One consequence of the Tietze extension theorem is that a subset of a metric space is $C$-embedded if and only if it is closed. Unlike the topological situation, a characterization of $U$-embedded subsets of metric spaces seems quite complicated. In this paper, we characterize those subsets of normed linear spaces which are $U$-embedded. As is usual in such situations, it is the convexity which will help us.

1. Preliminary definitions and results. Suppose that $(X, d)$ is a metric space. If $a, b \in X$, and $\varepsilon > 0$, then we say that $a$ and $b$ are $\varepsilon$-linked by $n$ links in $X$ if there exists $a = x_0, x_1, \ldots, x_n = b \in X$ such that $d(x_{k-1}, x_k) \leq \varepsilon$ for $k = 1, 2, \ldots, n$. The finite sequence $a = x_0, \ldots, x_n = b$ is called an $\varepsilon$-chain from $a$ to $b$. If there exists an $n$ such that $a$ and $b$ are $\varepsilon$-linked by $n$ links in $X$, we say that $a$ and $b$ are $\varepsilon$-linked in $X$. A metric space is uniformly connected if it is not the union of two nonempty subsets which are a positive distance apart. Clearly, every connected metric space is uniformly connected. It is not difficult to see that $X$ is uniformly connected if and only if for each $\varepsilon > 0$ and for each $a$ and $b$ in $X$, $a$ and $b$ are $\varepsilon$-linked in $X$.

Suppose $(X, d)$ is a metric space and $S$ is a subset such that every two elements of $S$ are $\varepsilon$-linked in $S$, where $\varepsilon$ is a positive number. Define

$$d_\varepsilon^S(a, b) = \inf \left\{ \sum_{i=1}^{m} d(z_{i-1}, z_i) : a = z_0, \ldots, z_m = b \text{ is an } \varepsilon\text{-chain in } S \right\}$$

and let

$$m_\varepsilon^S(a, b) = \min\{m : \text{there exists an } \varepsilon\text{-chain in } S \text{ from } a \text{ to } b \text{ having } m \text{ links}\}.$$

Then $d_\varepsilon^S(a, b)$ measures the shortest distance one has to travel between $a$ and $b$ given that each step taken is in $S$ and each step is at most $\varepsilon$ units long. On the
other hand, $m_{e}^{S}(a,b)$ gives the fewest number of steps of length at most $\varepsilon$ one must take to get from $a$ to $b$ provided that each step is in $S$. Fairly simple examples show that it is possible to have the sum of the distances along every $\varepsilon$-chain from $a$ to $b$ having $m_{e}^{S}(a,b)$ links be bounded away from $d_{e}^{S}(a,b)$. Now let

$$r_{e}^{S} = \sup\{d_{e}^{S}(a,b)/d(a,b) : a,b \in S, a \neq b\}$$

and let

$$r_{e}^{S} = \sup\{m_{e}^{S}(a,b)/m_{e}^{X}(a,b) : a,b \in S, a \neq b\}.$$

When the subspace $S$ is clear from the context, we omit the superscript $S$ in $r_{e}$ and $r_{e}^{S}$.

1.1. **PROPOSITION.** Suppose $(X,d)$ is a metric space and $S$ is a subspace of $X$. Then $r_{e} \leq 4r_{e} + 2$. Therefore, if $r_{e}$ is finite, then $r_{e}$ is finite.

**PROOF.** Suppose $a$ and $b$ are in $S$. Given $\delta > 0$, choose an $\varepsilon$-chain $o = z_{0}, z_{1}, \ldots, z_{N} = b$ from $a$ to $b$ such that

$$A = \sum_{i=1}^{N} d(z_{i-1}, z_{i}) \leq d_{e}^{S}(a,b) + \delta$$

where $N$ is the smallest integer for which there exists such an $\varepsilon$-chain. Then

$$(*) \ d(z_{2k-2}, z_{2k}) > \varepsilon \quad \text{for} \quad k = 1, \ldots, \lfloor N/2 \rfloor,$$

because if this inequality did not hold, we could get an $\varepsilon$-chain having fewer links and the corresponding $\varepsilon$-chain would give a sum of distances not exceeding $A$. From $(*)$, we get $A \geq \lfloor N/2 \rfloor \varepsilon / 2$, so

$$[N/2] \varepsilon / 2 \leq d_{e}^{S}(a,b) + \delta.$$ 

Set $m = m_{e}^{S}(a,b)$. By the choice of $N$ and the definition of $m_{e}^{S}(a,b)$, $m \leq N$. Repeated applications of the triangle inequality give $d(a,b) \leq \varepsilon m_{e}^{X}(a,b)$. Then we get

$$m_{e}^{S}(a,b)/m_{e}^{X}(a,b) = \varepsilon m_{e}^{S}(a,b)/\varepsilon m_{e}^{X}(a,b) \leq \varepsilon m_{e}^{S}(a,b)/d(a,b) = \varepsilon m/d(a,b) \leq N \varepsilon / d(a,b),$$

so

$$m_{e}^{S}(a,b)/m_{e}^{X}(a,b) \leq N \varepsilon / d(a,b) \leq [4(d_{e}^{S}(a,b) + \delta) + 2\varepsilon] / d(a,b) = 4d_{e}^{S}(a,b)/d(a,b) + (4\varepsilon + 2\delta) / d(a,b).$$

If $d(a,b) \geq \varepsilon$, then $m_{e}^{S}(a,b)/m_{e}^{X}(a,b) \leq 4r_{e} + (4\delta + 2\varepsilon) / \varepsilon = 4r_{e} + 2 + (4\delta / \varepsilon)$. On the other hand, if $d(a,b) < \varepsilon$, then $m_{e}^{S}(a,b) = m_{e}^{X}(a,b) = 1$ and $d_{e}^{S}(a,b) = d(a,b)$, so $m_{e}^{S}(a,b)/m_{e}^{X}(a,b) = 1 = d_{e}^{S}(a,b)/d(a,b)$. Therefore, $r_{e} \leq 4r_{e} + 2 + 4\delta / \varepsilon$ for each $\delta > 0$, so $r_{e} \leq 4r_{e} + 2$.

If $(X,d)$ is a metric space and $f : X \to R$ is a function, then $f$ is **Lipschitz for large distances** if for each $\varepsilon > 0$, there exists a constant $K$ (which will in general depend upon $\varepsilon$) such that $d(x,y) \geq \varepsilon$ implies that $|f(x) - f(y)| \leq K d(x,y)$. If $F$ is a family of functions from $X$ to $R$, then $F$ is said to be jointly **Lipschitz for large distances** if for each $\varepsilon > 0$ there exists a constant $K$ (depending upon $\varepsilon$) such that if $d(x,y) \geq \varepsilon$ and $f \in F$, then $|f(x) - f(y)| \leq K d(x,y)$. The phrase “$(X,d)$ is a normed linear space” will be used to mean that $d$ is the metric induced by the norm on the normed linear space $X$. 

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1.2. LEMMA [LR1]. If \((X,d)\) is a normed linear space and \(S\) is a subset of \(X\), then \(S\) is \(U\)-embedded in \(X\) if and only if each uniformly continuous function \(f : S \to R\) is Lipschitz for large distances.

1.3. LEMMA [LR2]. If \((X,d)\) is a normed linear space and \(S\) is a uniformly connected subset of \(X\), then \(S\) is \(U\)-embedded in \(X\) if and only if each equi-uniformly-continuous family \(F\) of functions from \(S\) to \(R\) is jointly Lipschitz for large distances.

2. The uniformly connected case. In this section, we give a characterization of those uniformly connected subsets of normed linear spaces which are \(U\)-embedded. In the next section we show how to modify the characterization for the case where the subset \(S\) is not assumed to be uniformly connected.

2.1. PROPOSITION. Suppose \((X,d)\) is a normed linear space, and suppose that \(S\) is a uniformly connected \(U\)-embedded subset of \(X\). Then \(r_\varepsilon\) is finite for each \(\varepsilon > 0\).

PROOF. Assume \(r_\varepsilon = +\infty\) for some \(\varepsilon > 0\). Then there exist sequences \((x_k)\) and \((y_k)\) of points of \(S\) such that \(x_k \neq y_k\) and \(d^S_\varepsilon(x_k,y_k) \geq kd(x_k,y_k)\) for \(k = 1,2,\ldots\). Since \(d(x,y) \leq \varepsilon\) implies that \(d^S_\varepsilon(x,y) = d(x,y)\), the choice of the \(x_k\)'s and \(y_k\)'s gives us that \(d(x_k,y_k) > \varepsilon\) for \(k \geq 2\). For \(k = 1,2,\ldots\), define \(g_k : S \to R\) by \(g_k(x) = d^S_\varepsilon(x,y_k)\). We claim that the family \(\{g_k : k = 1,2,\ldots\}\) is equi-uniformly continuous. Choose \(\eta > 0\) and let \(\delta = \min\{\eta,\varepsilon\}\). Suppose \(x,y \in S\) and \(d(x,y) < \delta\). Then \(d(x,y) < \varepsilon\). Given \(\rho > 0\), choose \(\varepsilon\)-chains \(y = y_0, y_1, \ldots, y_L(k) = y_k\) and \(x = x_0, x_1, \ldots, x_M(k) = y_k\) such that

\[
\sum_{i=1}^{L(k)} d(y_{i-1}, y_i) < d^S_\varepsilon(y,y_k) + \rho
\]

and

\[
\sum_{j=1}^{M(k)} d(x_{j-1}, x_j) < d^S_\varepsilon(x,y_k) + \rho.
\]

Then

\[
d^S_\varepsilon(x,y_k) \leq d(x,y) + \sum_{i=1}^{L(k)} d(y_{i-1}, y_i) < d(x,y) + d^S_\varepsilon(y,y_k) + \rho.
\]

Therefore,

\[
dx^S_\varepsilon(x,y_k) < d(x,y) < \eta.
\]

Similarly, one shows that

\[
d^S_\varepsilon(y,y_k) - d^S_\varepsilon(x,y_k) < d(x,y) < \eta.
\]

Therefore, \(|g_k(x) - g_k(y)| \leq \eta\). This proves the claim. However, \(|g_k(x_k) - g_k(y_k)| = d^S_\varepsilon(x_k,y_k) \geq kd(x_k,y_k), k = 1,2,\ldots\). Since \(d(x_k,y_k) > \varepsilon\) for \(k \geq 2\), this violates 1.3.

2.2. PROPOSITION. Suppose that \(S\) is a uniformly connected subset of a normed linear space \((X,d)\). If \(r_\varepsilon\) is finite for each positive \(\varepsilon\), then \(S\) is \(U\)-embedded in \(X\).

PROOF. Suppose \(f : S \to R\) is uniformly continuous. Suppose \(\varepsilon > 0\). We must find a constant \(K_\varepsilon\) such that \(d(x,y) \geq \varepsilon\) implies that \(|f(x) - f(y)| < K_\varepsilon d(x,y)|\).
Choose $\delta > 0$ such that $d(x, y) < \delta$ implies that $|f(x) - f(y)| < 1$. Assume $d(x, y) \geq \varepsilon$. Let $m = m_\delta(x, y)$ and let $x = x_0, x_1, \ldots, x_m = y$ be a $\delta$-chain in $S$. Then if we let $K_\varepsilon = r_\varepsilon[(\varepsilon + \delta)/\varepsilon \delta]$, we get

$$|f(x) - f(y)| \leq \sum_{i=1}^{m} |f(x_{i-1}) - f(x_i)|$$

$$\leq m = m_\delta(x, y)(m/m_\delta(x, y))$$

$$\leq r_\varepsilon[(d(x, y)/\delta) + 1] \quad \text{(because } m_\delta(x, y) = [(d(x, y)/\delta) + 1])$$

$$\leq r_\varepsilon\{d(x, y) + (\delta/\varepsilon)\varepsilon)/\delta$$

$$\leq r_\varepsilon\{(\varepsilon d(x, y) + \delta d(x, y))/\varepsilon \delta\} \quad \text{(since } \varepsilon \leq d(x, y))$$

$$= K_\varepsilon d(x, y).$$

Therefore, $f$ is Lipschitz for large distances.

Combining 1.1 with the preceding two propositions gives the following theorem:

**2.3. THEOREM.** Suppose that $S$ is a uniformly connected subset of the normed linear space $X$. Then the following are equivalent:

(i) $\mathcal{r}_\varepsilon$ is finite for each $\varepsilon > 0$.

(ii) $\mathcal{r}_\varepsilon$ is finite for each $\varepsilon > 0$.

(iii) $S$ is $U$-embedded in $X$.

**3. The general case.** In this section, we give a characterization of those subsets of normed linear spaces which are $U$-embedded. The characterization will use the characterization given in §2 for the case where the subspace is uniformly connected.

**3.1. PROPOSITION.** Suppose $S$ is a $U$-embedded subset of the normed linear space $(X, d)$. Then for each $\varepsilon > 0$ there exists a compact subset $F$ of $X$, which may be taken to be the union of finitely many line segments, such that

(i) every two elements of $S \cup F$ are $\varepsilon$-linked in $S \cup F$,

(ii) $\mathcal{r}_\varepsilon^{S \cup F}$, and therefore $\mathcal{r}_\varepsilon^{S \cup F}$, is finite.

**PROOF.** For each $p \in S$, define $C_p = \{x \in S: x$ and $p$ are $\varepsilon/3$-linked in $S\}$. If $C_p \neq C_q$, then $d(C_p, C_q) \geq \varepsilon/3$, so $\mathcal{C} = \{C_p: p \in S\}$ is a uniformly discrete family of nonempty subsets of $X$ whose union is the $U$-embedded set $S$. Therefore, $\mathcal{C}$ is finite. (See [LR$_1$].) Write $\mathcal{C} = \{D_0, D_1, \ldots, D_N\}$ and for each $k = 0, 1, \ldots, N$ choose $p_k \in D_k$. For $k = 1, 2, \ldots, N$, let $F_k$ be the line segment from $p_{k-1}$ to $p_k$ and let $F = \bigcup_{k=1}^{N} F_k$. With this definition, condition (i) is clearly satisfied. Now let $\hat{S} = \{p \in X: d(p, S \cup F) \leq \varepsilon/3\}$. It is easy to see that $\hat{S}$ is uniformly connected.

We claim that every uniformly continuous function $g: \hat{S} \to R$ which is identically zero on $S$ is bounded. It will then follow from [LR$_1$] that $\hat{S}$ is $U$-embedded in $X$. So assume $g: \hat{S} \to R$ is uniformly continuous and identically zero on $S$. Since $F$ is compact, it follows that the restriction of $g$ to $S \cup F$ is bounded. Let $K$ be an upper bound for the absolute value of this restriction. Now let $\delta > 0$ be such that $x, y \in S \cup F$, $d(x, y) < \delta$ imply $|g(x) - g(y)| \leq 1$. Given a point $x$ of $\hat{S}$, there exists a point $p$ of $S \cup F$ such that the segment from $x$ to $p$ is contained in $\hat{S}$ and has length at most $\varepsilon/3$. Then $|g(x)| \leq B + K$, where $B = [(\varepsilon/3\delta) + 1]$. Therefore, $\hat{S}$ is $U$-embedded in $X$. 

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It follows from 2.3 that \( r_{\varepsilon}^{S_\beta} \) is finite. We will show that this implies that \( r_{\varepsilon}^{S \cup F} \) is finite. Suppose \( a, b \in S \cup F \). Given \( \rho > 0 \), choose an \( \varepsilon/3 \)-chain \( a = a_0, a_1, \ldots, a_M = b \) in \( S \) such that

\[
A = \sum_{i=1}^{M} d(a_{i-1}, a_i) < d_{\varepsilon/3}^{S}(a, b) + \rho.
\]

We may assume without loss of generality that this \( \varepsilon/3 \) chain is minimal in the sense that \( d(a_0, a_2) > \varepsilon/3, d(a_2, a_4) > \varepsilon/3, \ldots \). (If this is not the case, inductively choose the elements of the chain so that the resulting chain is minimal and \((\#)\) will still hold.) Then at least half of the distances \( d(a_{i-1}, a_i) \) are at least \( \varepsilon/6 \), so

\[
A \geq (\varepsilon/6)[M/2], \quad \text{that is,} \quad M \leq (12A/\varepsilon) + 2.
\]

For each \( k = 1, 2, \ldots, M - 1 \), choose \( x_k \in S \cup F \) such that \( d(x_k, a_k) < \varepsilon/3 \). Then if \( x_0 = a \) and \( x_M = b \), one gets that \( d(x_{k-1}, x_k) \leq d(x_{k-1}, a_{k-1}) + d(a_{k-1}, a_k) + d(a_k, x_k) \leq d(a_{k-1}, a_k) + 2\varepsilon/3 \leq \varepsilon/3 + 2\varepsilon/3 = \varepsilon \), so \( a = x_0, x_1, \ldots, x_M = b \) is an \( \varepsilon \)-chain in \( S \cup F \). Furthermore,

\[
\sum_{i=1}^{M} d(x_{i-1}, x_i) \leq \sum_{i=1}^{M} d(a_{i-1}, a_i) + 2M\varepsilon/3.
\]

Therefore, using \((*)\) we get \( \sum_{i=1}^{M} d(x_{i-1}, x_i) \leq A + (2\varepsilon/3)[(12A/\varepsilon) + 2] = 9A + (4\varepsilon/3) \leq 9d_{\varepsilon/3}^{S}(a, b) + [9\rho + (4\varepsilon/3)] \). Therefore, \( d_{\varepsilon}^{S \cup F}(a, b) \leq 9d_{\varepsilon/3}^{S}(a, b) + (4\varepsilon/3) \). Hence, \( r_{\varepsilon}^{S \cup F} \leq 9r_{\varepsilon/3}^{S} + (4\varepsilon/3) \).

3.2. PROPOSITION. Suppose \( S \) is a subset of a normed linear space \( X \). Suppose that for each \( \varepsilon > 0 \) there exists a compact set \( F \) such that

(i) every two elements of \( S \cup F \) are \( \varepsilon \)-linked in \( S \cup F \),

(ii) \( r_{\varepsilon}^{S \cup F} \) is finite.

Then \( S \) is \( U \)-embedded in \( X \).

PROOF. Suppose \( f : S \to R \) is uniformly continuous. By a theorem of Isbell [I], there exists an \( \varepsilon > 0 \) such that \( f \) can be extended to a uniformly continuous function \( f_1 : S_\varepsilon \to R \), where \( S_\varepsilon = \{ x \in X : d(x, S) \leq \varepsilon \} \). Choose a positive \( \delta < \varepsilon \) such that \( x, y \in S_\varepsilon \) and \( d(x, y) < \delta \) imply that \( |f_1(x) - f_1(y)| < 1 \). Let \( F \) be the compact set given by the hypothesis corresponding to \( \delta \). By [LR1], \( f_1 \) can be extended to a uniformly continuous function \( \hat{f} : S_\varepsilon \cup F \to R \). We claim that there exists a constant \( C \) such that if \( x \) and \( y \) are elements of \( S \cup F \) satisfying \( d(x, y) \leq \delta \), then \( |\hat{f}(x) - \hat{f}(y)| \leq C \). Let \( M \) be a constant such that \( |f(x)| < M \) for all \( x \in F \). Suppose \( x, y \in S \cup F \) and \( d(x, y) < \delta \). If \( \{ x, y \} \subseteq F \), then \( |f(x) - f(y)| \leq 2M \). If \( \{ x, y \} \subseteq S \), then \( |f(x) - f(y)| < 1 \). If \( x \in S, y \in F \), then \( y \in S_\varepsilon \) (because \( d(x, y) < \varepsilon \)) so again we have \( |f(x) - f(y)| < 1 \). Therefore, we may choose \( C = 2M + 1 \).

Now assume \( d(x, y) \geq \varepsilon \), where \( x \) and \( y \) are points of \( S \cup F \). Let \( m = m_{\varepsilon}^{S \cup F}(x, y) \) and let \( x = x_0, x_1, \ldots, x_m = y \) be a \( \delta \)-chain in \( S \cup F \). Then

\[
|f(x) - f(y)| \leq \sum_{i=1}^{m} |f(x_{i-1}) - f(x_i)| \leq mC
\]

\[
\leq Cr_{\varepsilon}^{S \cup F}[(\varepsilon + \delta)/\varepsilon\delta]d(x, y) = Ke d(x, y),
\]
where $K_\varepsilon = C r_{\delta}^{S \cup F}(\varepsilon + \delta)/\varepsilon \delta$. Therefore, $f$ is Lipschitz for large distances.

Combining 3.1 and 3.2 gives the following theorem:

3.3. **THEOREM.** Suppose $S$ is a subset of the normed linear space $(X, d)$. Then the following are equivalent:

(i) $S$ is $U$-embedded in $X$,

(ii) for each $\varepsilon > 0$, there exists a compact set $F$ (which may be taken to be a union of finitely many line segments) such that any two elements of $S \cup F$ are $\varepsilon$-linked in $S \cup F$ and such that $r_{\varepsilon}^{S \cup F}$ is finite.

**REMARKS.** 1. If $(X, d)$ is any uniformly connected metric space, then by embedding $X$ isometrically in a normed linear space and appealing to 1.2 and 2.3 one sees that if every uniformly continuous function $f: X \to \mathbb{R}$ is Lipschitz for large distances, then $r_{\varepsilon}^X$ is finite for each positive $\varepsilon$. In fact, the converse of this statement is true as well: If $r_{\varepsilon}^X$ is finite for each positive $\varepsilon$, then every uniformly continuous function $f: X \to \mathbb{R}$ is Lipschitz for large distances. Since there are easy examples of metric spaces where not every uniformly continuous real-valued function is Lipschitz for large distances, this means that the finiteness of $r_{\varepsilon}^S$ for each positive $\varepsilon$ is not in general equivalent to the $U$-embedding of a subspace $S$ of a space $X$.

2. We do not know an example of a metric space $(X, d)$ and a non-$U$-embedded uniformly connected subset $S$ such that $r_{\varepsilon}^S$ is finite for each positive $\varepsilon$. However, the following example shows that a uniformly connected, $U$-embedded subset of a metric space can have infinite $r_{\varepsilon}^S$ for all sufficiently small $\varepsilon > 0$. 

![Diagram](http://www.ams.org/journal-terms-of-use)
3.4. EXAMPLE. There exists a (uniformly) connected metric space $X \subset \mathbb{R}^2$ and a $U$-embedded (uniformly) connected subset $S$ such that $r_\varepsilon = +\infty$ for all positive $\varepsilon < 1/2$. Rather than give a description of $X$ and $S$, we will draw pictures. Any uniformly continuous $f: S \rightarrow \mathbb{R}$ can be extended to a uniformly continuous function $F: X \rightarrow \mathbb{R}$ by making $F$ constant on the small vertical whiskers growing along the $y$-axis. By restricting our attention to those horizontal bands of $S$ of height $1/n$, one can show that $r_{1/(2n)} = \infty$ for $n = 1, 2, \ldots$. For an arbitrary positive $\varepsilon \leq 1/2$, find an $n$ satisfying $1/(2n) < \varepsilon < 1/n$ (that is, $1/2 < n\varepsilon \leq 1$), and by restricting our attention to the horizontal bands of height $1/n$, we can again show that $r_\varepsilon = \infty$.

REMARKS. 1. By modifying the construction in 3.4, it is possible to find a connected metric space $X$ and a connected $U$-embedded subset $S$ such that $r_\varepsilon = \infty$ for all $\varepsilon > 0$.

2. By appealing to the results in [LR$_3$] the results in this paper also give results about Hilbert spaces. For example, it is shown in [LR$_3$] that a subset $S$ of a Hilbert space $H$ is $U$-embedded in $H$ if and only if every uniformly continuous function $f: S \rightarrow H$ extends to a uniformly continuous function $F: H \rightarrow H$. Thus, we also have characterized those subsets of Hilbert space for which every uniformly continuous function into a Hilbert space extends to a uniformly continuous function.

REFERENCES


