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U-EMBEDDED SUBSETS OF NORMED LINEAR SPACES

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ABSTRACT. A subset S of a metric space X is U-embedded in X if every uniformly continuous function $f\colon S \to R$ extends to a uniformly continuous function $F\colon X \to R$. Thus U-embedding is the uniform analogue of C-embedding. The Tietze extension theorem tells us exactly which subsets of metric spaces are C-embedded. The uniform analogue would tell us exactly which subsets of metric spaces are U-embedded. In this paper, a characterization of U-embedded subsets of the Euclidean plane (or any normed linear space) is given.

A subset S of a uniform space X is U-embedded in X if every real-valued uniformly continuous function $f\colon S \to R$ extends to a uniformly continuous function $F\colon X \to R$. Thus, U-embedding is the uniform analogue of C-embedding in topological spaces. One consequence of the Tietze extension theorem is that a subset of a metric space is C-embedded if and only if it is closed. Unlike the topological situation, a characterization of U-embedded subsets of metric spaces seems quite complicated. In this paper, we characterize those subsets of normed linear spaces which are U-embedded. As is usual in such situations, it is the convexity which will help us.

1. Preliminary definitions and results. Suppose that (X, d) is a metric space. If $a, b \in X$, and $\varepsilon > 0$, then we say that a and b are *ε-linked by n links in X* if there exists $a = x_0, x_1, \ldots, x_n = b \in X$ such that $d(x_{k-1}, x_k) \leq \varepsilon$ for $k = 1, 2, \ldots, n$. The finite sequence $a = x_0, \ldots, x_n = b$ is called an *ε-chain from a to b*. If there exists an n such that a and b are ε-linked by n links in X, we say that a and b are *ε-linked in X*. A metric space is *uniformly connected* if it is not the union of two nonempty subsets which are a positive distance apart. Clearly, every connected metric space is uniformly connected. It is not difficult to see that X is uniformly connected if and only if for each $\varepsilon > 0$ and for each a and b in X, a and b are ε-linked in X.

Suppose (X, d) is a metric space and S is a subset such that every two elements of S are ε-linked in S, where ε is a positive number. Define

$$d_\varepsilon^S(a, b) = \inf\left\{\sum_{i=1}^{m} d(z_{i-1}, z_i)\colon a = z_0, \ldots, z_m = b \text{ is an } \varepsilon\text{-chain in } S\right\}$$

and let

$$m_\varepsilon^S(a, b) = \min\{m\colon \text{ there exists an } \varepsilon\text{-chain in } S \text{ from } a \text{ to } b \text{ having } m \text{ links}\}.$$

Then $d_\varepsilon^S(a, b)$ measures the shortest distance one has to travel between a and b given that each step taken is in S and each step is at most ε units long. On the

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other hand, $m_\varepsilon^S(a,b)$ gives the fewest number of steps of length at most ε one must take to get from a to b provided that each step is in S. Fairly simple examples show that it is possible to have the sum of the distances along every ε-chain from a to b having $m_\varepsilon^S(a,b)$ links be bounded away from $d_\varepsilon^S(a,b)$. Now let

$$\hat{r}_\varepsilon^S = \sup\{d_\varepsilon^S(a,b)/d(a,b) \colon a, b \in S, a \neq b\}$$

and let

$$r_\varepsilon^S = \sup\{m_\varepsilon^S(a,b)/m_\varepsilon^X(a,b) \colon a, b \in S, a \neq b\}.$$

When the subspace S is clear from the context, we omit the superscript S in $\hat{r}_\varepsilon^S$ and r_ε^S.

1.1. PROPOSITION. *Suppose (X,d) is a metric space and S is a subspace of X. Then $r_\varepsilon \leq 4\hat{r}_\varepsilon + 2$. Therefore, if $\hat{r}_\varepsilon$ is finite, then r_ε is finite.*

PROOF. Suppose a and b are in S. Given $\delta > 0$, choose an ε-chain $a = z_0, z_1, \ldots, z_N = b$ from a to b such that

$$A = \sum_{i=1}^{N} d(z_{i-1}, z_i) \leq d_\varepsilon^S(a,b) + \delta$$

where N is the smallest integer for which there exists such an ε-chain. Then

$$(*) \qquad d(z_{2k-2}, z_{2k}) > \varepsilon \quad \text{for } k = 1, \ldots, [N/2],$$

because if this inequality did not hold, we could get an ε-chain having fewer links and the corresponding ε-chain would give a sum of distances not exceeding A. From $(*)$, we get $A \geq [N/2]\varepsilon/2$, so

$$[N/2]\varepsilon/2 \leq d_\varepsilon^S(a,b) + \delta.$$

Set $m = m_\varepsilon^S(a,b)$. By the choice of N and the definition of $m_\varepsilon^S(a,b)$, $m \leq N$. Repeated applications of the triangle inequality give $d(a,b) \leq \varepsilon m_\varepsilon^X(a,b)$. Then we get

$$\begin{aligned} m_\varepsilon^S(a,b)/m_\varepsilon^X(a,b) &= \varepsilon m_\varepsilon^S(a,b)/\varepsilon m_\varepsilon^X(a,b) \\ &\leq \varepsilon m_\varepsilon^S(a,b)/d(a,b) = \varepsilon m/d(a,b) \leq N\varepsilon/d(a,b), \end{aligned}$$

so

$$\begin{aligned} m_\varepsilon^S(a,b)/m_\varepsilon^X(a,b) \leq N\varepsilon/d(a,b) &\leq [4(d_\varepsilon^S(a,b) + \delta) + 2\varepsilon]/d(a,b) \\ &= 4d_\varepsilon^S(a,b)/d(a,b) + (4\delta + 2\varepsilon)/d(a,b). \end{aligned}$$

If $d(a,b) \geq \varepsilon$, then $m_\varepsilon^S(a,b)/m_\varepsilon^X(a,b) \leq 4\hat{r}_\varepsilon + (4\delta + 2\varepsilon)/\varepsilon = 4\hat{r}_\varepsilon + 2 + (4\delta/\varepsilon)$. On the other hand, if $d(a,b) < \varepsilon$, then $m_\varepsilon^S(a,b) = m_\varepsilon^X(a,b) = 1$ and $d_\varepsilon^S(a,b) = d(a,b)$, so $m_\varepsilon^S(a,b)/m_\varepsilon^X(a,b) = 1 = d_\varepsilon^S(a,b)/d(a,b)$. Therefore, $r_\varepsilon \leq 4\hat{r}_\varepsilon + 2 + 4\delta/\varepsilon$ for each $\delta > 0$, so $r_\varepsilon \leq 4\hat{r}_\varepsilon + 2$.

If (X,d) is a metric space and $f \colon X \to R$ is a function, then f is *Lipschitz for large distances* if for each $\varepsilon > 0$, there exists a constant K (which will in general depend upon ε) such that $d(x,y) \geq \varepsilon$ implies that $|f(x) - f(y)| \leq Kd(x,y)$. If F is a family of functions from X to R, then F is said to be *jointly Lipschitz for large distances* if for each $\varepsilon > 0$ there exists a constant K (depending upon ε) such that if $d(x,y) \geq \varepsilon$ and $f \in F$, then $|f(x) - f(y)| \leq Kd(x,y)$. The phrase "$(X,d)$ is a normed linear space" will be used to mean that d is the metric induced by the norm on the normed linear space X.

1.2. LEMMA [$\mathbf{LR}_1$]. *If (X,d) is a normed linear space and S is a subset of X, then S is U-embedded in X if and only if each uniformly continuous function $f\colon S \to R$ is Lipschitz for large distances.*

1.3. LEMMA [$\mathbf{LR}_2$]. *If (X,d) is a normed linear space and S is a uniformly connected subset of X, then S is U-embedded in X if and only if each equi-uniformly-continuous family F of functions from S to R is jointly Lipschitz for large distances.*

2. The uniformly connected case. In this section, we give a characterization of those uniformly connected subsets of normed linear spaces which are U-embedded. In the next section we show how to modify the characterization for the case where the subset S is not assumed to be uniformly connected.

2.1. PROPOSITION. *Suppose (X,d) is a normed linear space, and suppose that S is a uniformly connected U-embedded subset of X. Then $\hat{r}_\varepsilon$ is finite for each $\varepsilon > 0$.*

PROOF. Assume $\hat{r}_\varepsilon = +\infty$ for some $\varepsilon > 0$. Then there exist sequences (x_k) and (y_k) of points of S such that $x_k \neq y_k$ and $d_\varepsilon^S(x_k, y_k) \geq kd(x_k, y_k)$ for $k = 1, 2, \ldots$. Since $d(x,y) \leq \varepsilon$ implies that $d_\varepsilon^S(x,y) = d(x,y)$, the choice of the x_k's and y_k's gives us that $d(x_k, y_k) > \varepsilon$ for $k \geq 2$. For $k = 1, 2, \ldots$, define $g_k\colon S \to R$ by $g_k(x) = d_\varepsilon^S(x, y_k)$. We claim that the family $\{g_k\colon k = 1, 2, \ldots\}$ is equi-uniformly continuous. Choose $\eta > 0$ and let $\delta = \min\{\eta, \varepsilon\}$. Suppose $x, y \in S$ and $d(x,y) < \delta$. Then $d(x,y) \leq \varepsilon$. Given $\rho > 0$, choose ε-chains $y = y_0, y_1, \ldots, y_{L(k)} = y_k$ and $x = x_0, x_1, \ldots, x_{M(k)} = y_k$ such that

$$\sum_{i=1}^{L(k)} d(y_{i-1}, y_i) < d_\varepsilon^S(y, y_k) + \rho$$

and

$$\sum_{j=1}^{M(k)} d(x_{j-1}, x_j) < d_\varepsilon^S(x, y_k) + \rho.$$

Then

$$d_\varepsilon^S(x, y_k) \leq d(x,y) + \sum_{i=1}^{L(k)} d(y_{i-1}, y_i) < d(x,y) + d_\varepsilon^S(y, y_k) + \rho.$$

Therefore,

$$d_\varepsilon^S(x, y_k) - d_\varepsilon^S(y, y_k) < d(x,y) < \eta.$$

Similarly, one shows that

$$d_\varepsilon^S(y, y_k) - d_\varepsilon^S(x, y_k) < d(x,y) < \eta.$$

Therefore, $|g_k(x) - g_k(y)| \leq \eta$. This proves the claim. However, $|g_k(x_k) - g_k(y_k)| = d_\varepsilon^S(x_k, y_k) \geq kd(x_k, y_k)$, $k = 1, 2, \ldots$. Since $d(x_k, y_k) > \varepsilon$ for $k \geq 2$, this violates 1.3.

2.2. PROPOSITION. *Suppose that S is a uniformly connected subset of a normed linear space (X,d). If r_ε is finite for each positive ε, then S is U-embedded in X.*

PROOF. Suppose $f\colon S \to R$ is uniformly continuous. Suppose $\varepsilon > 0$. We must find a constant K_ε such that $d(x,y) \geq \varepsilon$ implies that $|f(x) - f(y)| < K_\varepsilon d(x,y)$.

Choose $\delta > 0$ such that $d(x,y) < \delta$ implies that $|f(x) - f(y)| < 1$. Assume $d(x,y) \geq \varepsilon$. Let $m = m_\delta^S(x,y)$ and let $x = x_0, x_1, \ldots, x_m = y$ be a δ-chain in S. Then if we let $K_\varepsilon = r_\delta[(\varepsilon + \delta)/\varepsilon\delta]$, we get

$$\begin{aligned}
|f(x) - f(y)| &\leq \sum_{i=1}^{m} |f(x_{i-1}) - f(x_i)| \\
&\leq m = m_\delta^X(x,y)(m/m_\delta^X(x,y)) \\
&\leq r_\delta[(d(x,y)/\delta) + 1] \quad (\text{because } m_\delta^X(x,y) = [(d(x,y)/\delta) + 1]) \\
&\leq r_\delta\{d(x,y) + (\delta/\varepsilon)\varepsilon\}/\delta \\
&\leq r_\delta\{(\varepsilon d(x,y) + \delta d(x,y))/\varepsilon\delta\} \quad (\text{since } \varepsilon \leq d(x,y)) \\
&= K_\varepsilon d(x,y).
\end{aligned}$$

Therefore, f is Lipschitz for large distances.

Combining 1.1 with the preceding two propositions gives the following theorem:

2.3. THEOREM. *Suppose that S is a uniformly connected subset of the normed linear space X. Then the following are equivalent:*

(i) *$\hat{r}_\varepsilon$ is finite for each $\varepsilon > 0$.*

(ii) *r_ε is finite for each $\varepsilon > 0$.*

(iii) *S is U-embedded in X.*

3. The general case. In this section, we give a characterization of those subsets of normed linear spaces which are U-embedded. The characterization will use the characterization given in §2 for the case where the subspace is uniformly connected.

3.1. PROPOSITION. *Suppose S is a U-embedded subset of the normed linear space (X,d). Then for each $\varepsilon > 0$ there exists a compact subset F of X, which may be taken to be the union of finitely many line segments, such that*

(i) *every two elements of $S \cup F$ are ε-linked in $S \cup F$,*

(ii) *$\hat{r}_\varepsilon^{S\cup F}$, and therefore $r_\varepsilon^{S\cup F}$, is finite.*

PROOF. For each $p \in S$, define $C_p = \{x \in S: x$ and p are $\varepsilon/3$-linked in $S\}$. If $C_p \neq C_q$, then $d(C_p, C_q) \geq \varepsilon/3$, so $\mathcal{C} = \{C_p: p \in S\}$ is a uniformly discrete family of nonempty subsets of X whose union is the U-embedded set S. Therefore, $\mathcal{C}$ is finite. (See [$\mathbf{LR}_1$].) Write $\mathcal{C} = \{D_0, D_1, \ldots, D_N\}$ and for each $k = 0, 1, \ldots, N$ choose $p_k \in D_k$. For $k = 1, 2, \ldots, N$, let F_k be the line segment from p_{k-1} to p_k and let $F = \bigcup_{k=1}^N F_k$. With this definition, condition (i) is clearly satisfied. Now let $\hat{S} = \{p \in X: d(p, S \cup F) \leq \varepsilon/3\}$. It is easy to see that $\hat{S}$ is uniformly connected. We claim that every uniformly continuous function $g: \hat{S} \to R$ which is identically zero on S is bounded. It will then follow from [$\mathbf{LR}_1$] that $\hat{S}$ is U-embedded in X. So assume $g: \hat{S} \to R$ is uniformly continuous and identically zero on S. Since F is compact, it follows that the restriction of g to $S \cup F$ is bounded. Let K be an upper bound for the absolute value of this restriction. Now let $\delta > 0$ be such that $x, y \in S \cup F$, $d(x,y) < \delta$ imply $|g(x) - g(y)| \leq 1$. Given a point x of $\hat{S}$, there exists a point p of $S \cup F$ such that the segment from x to p is contained in $\hat{S}$ and has length at most $\varepsilon/3$. Then $|g(x)| \leq B + K$, where $B = [(\varepsilon/3\delta) + 1]$. Therefore, $\hat{S}$ is U-embedded in X.

It follows from 2.3 that $\hat{r}^S_{\varepsilon/3}$ is finite. We will show that this implies that $\hat{r}^{S\cup F}_\varepsilon$ is finite. Suppose $a, b \in S \cup F$. Given $\rho > 0$, choose an $\varepsilon/3$-chain $a = a_0, a_1, \dots, a_M = b$ in $\hat{S}$ such that

$$(\#) \qquad A = \sum_{i=1}^{M} d(a_{i-1}, a_k) < d^S_{\varepsilon/3}(a, b) + \rho.$$

We may assume without loss of generality that this $\varepsilon/3$ chain is minimal in the sense that $d(a_0, a_2) > \varepsilon/3$, $d(a_2, a_4) > \varepsilon/3, \dots$. (If this is not the case, inductively choose the elements of the chain so that the resulting chain is minimal and (#) will still hold.) Then at least half of the distances $d(a_{i-1}, a_i)$ are at least $\varepsilon/6$, so

$$(*) \qquad A \geq (\varepsilon/6)[M/2], \quad \text{that is,} \quad M \leq (12A/\varepsilon) + 2.$$

For each $k = 1, 2, \dots, M-1$, choose $x_k \in S \cup F$ such that $d(x_k, a_k) < \varepsilon/3$. Then if $x_0 = a$ and $x_M = b$, one gets that $d(x_{k-1}, x_k) \leq d(x_{k-1}, a_{k-1}) + d(a_{k-1}, a_k) + d(a_k, x_k) \leq d(a_{k-1}, a_k) + 2\varepsilon/3 \leq \varepsilon/3 + 2\varepsilon/3 = \varepsilon$, so $a = x_0, x_1, \dots, x_M = b$ is an ε-chain in $S \cup F$. Furthermore,

$$\sum_{i=1}^{M} d(x_{i-1}, x_i) \leq \sum_{i=1}^{M} d(a_{i-1}, a_i) + 2M\varepsilon/3.$$

Therefore, using $(*)$ we get $\sum_{i=1}^{M} d(x_{i-1}, x_i) \leq A + (2\varepsilon/3)[(12A/\varepsilon) + 2] = 9A + (4\varepsilon/3) \leq 9d^S_{\varepsilon/3}(a, b) + [9\rho + (4\varepsilon/3)]$. Therefore, $d^{S\cup F}_\varepsilon(a, b) \leq 9d^S_{\varepsilon/3}(a, b) + (4\varepsilon/3)$. Hence, $\hat{r}^{S\cup F}_\varepsilon \leq 9\hat{r}^S_{\varepsilon/3} + (4\varepsilon/3)$.

3.2. PROPOSITION. *Suppose S is a subset of a normed linear space X. Suppose that for each $\varepsilon > 0$ there exists a compact set F such that*

(i) *every two elements of $S \cup F$ are ε-linked in $S \cup F$,*

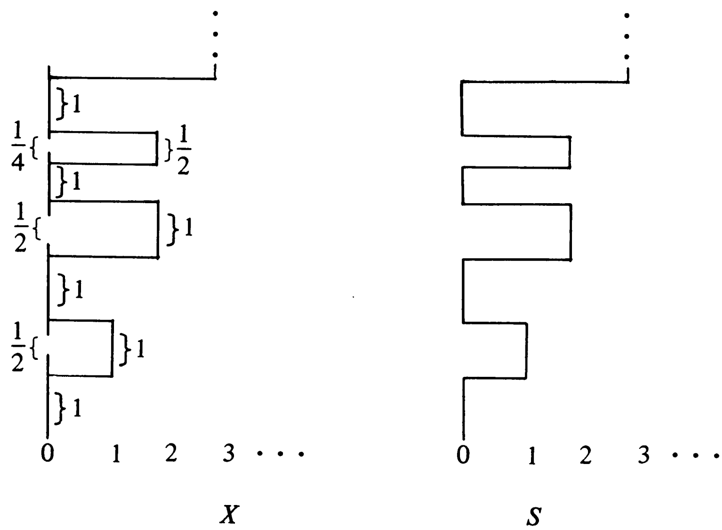
(ii) *$r^{S\cup F}_\varepsilon$ is finite.*

Then S is U-embedded in X.

PROOF. Suppose $f\colon S \to R$ is uniformly continuous. By a theorem of Isbell [**I**], there exists an $\varepsilon > 0$ such that f can be extended to a uniformly continuous function $f_1\colon S_\varepsilon \to R$, where $S_\varepsilon = \{x \in X\colon d(x, S) \leq \varepsilon\}$. Choose a positive $\delta < \varepsilon$ such that $x, y \in S_\varepsilon$ and $d(x, y) < \delta$ imply that $|f_1(x) - f_1(y)| < 1$. Let F be the compact set given by the hypothesis corresponding to δ. By [$\mathbf{LR}_1$], f_1 can be extended to a uniformly continuous function $\hat{f}\colon S_\varepsilon \cup F \to R$. We claim that there exists a constant C such that if x and y are elements of $S \cup F$ satisfying $d(x, y) < \delta$, then $|\hat{f}(x) - \hat{f}(y)| \leq C$. Let M be a constant such that $|f(x)| < M$ for all x in F. Suppose $x, y \in S \cup F$ and $d(x, y) < \delta$. If $\{x, y\} \subseteq F$, then $|f(x) - f(y)| \leq 2M$. If $\{x, y\} \subseteq S$, then $|f(x) - f(y)| < 1$. If $x \in S$, $y \in F$, then $y \in S_\varepsilon$ (because $d(x, y) < \varepsilon$) so again we have $|f(x) - f(y)| < 1$. Therefore, we may choose $C = 2M + 1$.

Now assume $d(x, y) \geq \varepsilon$, where x and y are points of $S \cup F$. Let $m = m^{S\cup F}_\delta(x, y)$ and let $x = x_0, x_1, \dots, x_m = y$ be a δ-chain in $S \cup F$. Then

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{i=1}^{m} |f(x_{i-1}) - f(x_i)| \leq mC \\ &\leq Cr^{S\cup F}_\delta[(\varepsilon + \delta)/\varepsilon\delta] d(x, y) = K_\varepsilon d(x, y), \end{aligned}$$



Diagram

where $K_\varepsilon = Cr_\delta^{S\cup F}(\varepsilon+\delta)/\varepsilon\delta$. Therefore, f is Lipschitz for large distances.

Combining 3.1 and 3.2 gives the following theorem:

3.3. THEOREM. *Suppose S is a subset of the normed linear space (X, d). Then the following are equivalent*:

(i) *S is U-embedded in X,*

(ii) *for each $\varepsilon > 0$, there exists a compact set F (which may be taken to be a union of finitely many line segments) such that any two elements of $S \cup F$ are ε-linked in $S \cup F$ and such that $r_\varepsilon^{S\cup F}$ is finite.*

REMARKS. 1. If (X, d) is any uniformly connected metric space, then by embedding X isometrically in a normed linear space and appealing to 1.2 and 2.3 one sees that if every uniformly continuous function $f\colon X \to R$ is Lipschitz for large distances, then r_ε^X is finite for each positive ε. In fact, the converse of this statement is true as well: If r_ε^X is finite for each positive ε, then every uniformly continuous function $f\colon X \to R$ is Lipschitz for large distances. Since there are easy examples of metric spaces where not every uniformly continuous real-valued function is Lipschitz for large distances, this means that the finiteness of r_ε^S for each positive ε is not in general equivalent to the U-embedding of a subspace S of a space X.

2. We do not know an example of a metric space (X, d) and a non-U-embedded uniformly connected subset S such that r_ε^S is finite for each positive ε. However, the following example shows that a uniformly connected, U-embedded subset of a metric space can have infinite r_ε^S for all sufficiently small $\varepsilon > 0$.

3.4. EXAMPLE. There exists a (uniformly) connected metric space $X \subset R^2$ and a U-embedded (uniformly) connected subset S such that $r_\varepsilon = +\infty$ for all positive $\varepsilon < 1/2$. Rather than give a description of X and S, we will draw pictures. Any uniformly continuous $f: S \to R$ can be extended to a uniformly continuous function $F: X \to R$ by making F constant on the small vertical whiskers growing along the y-axis. By restricting our attention to those horizontal bands of S of height $1/n$, one can show that $r_{1/(2n)} = \infty$ for $n = 1, 2, \ldots$. For an arbitrary positive $\varepsilon \leq 1/2$, find an n satisfying $1/(2n) < \varepsilon < 1/n$ (that is, $1/2 < n\varepsilon \leq 1$), and by restricting our attention to the horizontal bands of height $1/n$, we can again show that $r_\varepsilon = \infty$.

REMARKS. 1. By modifying the construction in 3.4, it is possible to find a connected metric space X and a connected U-embedded subset S such that $r_\varepsilon = \infty$ for all $\varepsilon > 0$.

2. By appealing to the results in [**LR**$_3$] the results in this paper also give results about Hilbert spaces. For example, it is shown in [**LR**$_3$] that a subset S of a Hilbert space H is U-embedded in H if and only if every uniformly continuous function $f: S \to H$ extends to a uniformly continuous function $F: H \to H$. Thus, we also have characterized those subsets of Hilbert space for which every uniformly continuous function into a Hilbert space extends to a uniformly continuous function.

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