WHITNEY LEVELS IN $C_p(X)$ ARE ARS

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ABSTRACT. For $X$ a metric continuum, and $p \in X$, we show that the Whitney levels in the relative hyperspace $C_p(X) = \{ K \in C(X) | p \in K \}$ are absolute retracts.

1. Introduction. For $(X, d)$ a metric continuum, let $C(X)$ denote the hyperspace of subcontinua with the Hausdorff metric $H$. A Whitney map $\mu: C(X) \to [0, 1]$ is a map such that $\mu(\{x\}) = 0$ for each $x \in X$, $\mu(X) = 1$, and $\mu(K) < \mu(M)$ whenever $K \subsetneq M$. Such maps may always be constructed [8]. The point-inverses $\mu^{-1}(t)$, $0 \leq t \leq 1$, are subcontinua in $C(X)$ [4], and are called Whitney levels. For $p \in X$, let $C_p(X) = \{ K \in C(X) | p \in K \}$ be the relative hyperspace, and let $\mu_p$ denote the restriction of $\mu$ to $C_p(X)$. Eberhart [3] showed that $C_p(X)$ is always an AR. In this paper we consider the relative Whitney levels $\mu_p^{-1}(t)$ in $C_p(X)$. We will use a construction in a space of order arcs to show that each Whitney level $\mu_p^{-1}(t)$ is an AR.

Krasinkiewicz and Nadler [7] and Rogers [11] have shown that $\mu_p^{-1}(t)$ is arcwise connected, and Rogers [10] showed that $\mu_p^{-1}(t)$ is acyclic. We point out that the Whitney levels $\mu_p^{-1}(t)$ need not be arcwise connected [6], and even if $X$ is a 2-cell, they need not be ARs [9].

2. The space of order arcs $\Lambda_p(X)$. An arc $\alpha \subset C(X)$ is an order arc if, for all $K, M \in \alpha$, either $K \subset M$ or $M \subset K$. Then $\bigcap \alpha = \bigcap \{M | M \in \alpha\}$ and $\bigcup \alpha = \bigcup \{M | M \in \alpha\}$ are the endpoints of $\alpha$. For every pair $A, B \subset C(X)$ with $A \subset B$, there exists an order arc $\alpha$ with $\bigcap \alpha = A$ and $\bigcup \alpha = B$ [6]. An order arc $\alpha$ may be parametrized by defining $\alpha(t)$ to be the unique $K \in \alpha$ such that $\mu(K) = (1-t) \cdot \mu(\bigcap \alpha) + t \cdot \mu(\bigcup \alpha)$. Let $\Lambda_p$ denote the space of maximal order arcs in $C_p(X)$, i.e., $\Lambda_p = \{ \text{order arcs } \alpha \subset C_p(X) | \alpha(0) = \{p\} \text{ and } \alpha(1) = X \subset C(C(X)) \}$. For each $t \in [0, 1]$, the evaluation map $e_t: \Lambda_p \to \mu_p^{-1}(t)$, defined by $e_t(\alpha) = \alpha(t)$, is onto. We will show that $\mu_p^{-1}(t)$ is an absolute extensor for metric spaces by imitating Dugundji’s proof of the extension property for maps into locally convex linear metric spaces [2]. To do this, we utilize the evaluation map $e_t$ and a type of convex structure on $\Lambda_p$.

For $\alpha_1, \ldots, \alpha_n \in \Lambda_p$ and $t_1, \ldots, t_{n-1} \in [0, 1]$, let $\beta = \langle \alpha_1, t_1; \ldots; \alpha_{n-1}, t_{n-1}; \alpha_n \rangle$ be the element of $\Lambda_p$ defined by

$$\beta = \{ \alpha_1(s) | 0 \leq s \leq t_1 \} \cup \{ \alpha_1(t_1) \cup \alpha_2(s) | 0 \leq s \leq t_2 \}$$
$$\cdots \cup \{ \alpha_1(t_1) \cup \cdots \cup \alpha_{n-1}(t_{n-1}) \cup \alpha_n(s) | 0 \leq s \leq 1 \}.$$
This construction is continuous, in the sense that if \( \alpha_m^m \to \alpha_i \) and \( t_m^i \to t_i \) as \( m \to \infty \), for each \( i \), then \( (\alpha_1^m, t_1^m; \ldots; \alpha_{n-1}^m, t_{n-1}^m; \alpha_n^m) \to (\alpha_1, t_1; \ldots; \alpha_{n-1}, t_{n-1}; \alpha_n) \).

3. Extending maps into \( \mu_p^{-1}(t) \). Let \( (Z, \rho) \) be a metric space, and \( A \subset Z \) a closed subset. Given a map \( g: A \to \mu_p^{-1}(t) \), we define an extension \( \hat{g}: Z \to \mu_p^{-1}(t) \) of \( g \) by a Dugundji-type construction. For every \( x \in Z - A \), let \( B_x = \{ z \in Z | \rho(x, z) < 1/2 \cdot \rho(x, A) \} \). Let \( U = \{ U_\alpha | \alpha \in A \} \) be a neighborhood finite open refinement of \( \{ B_x | x \in Z - A \} \), indexed by a well-ordered set \( A \). Let \( \{ \phi_\alpha | \alpha \in A \} \) be a partition of unity of \( Z - A \) subordinated to \( U \). With each \( \alpha \in A \), associate \( a_\alpha \in A \) as follows: Choose \( x_\alpha \in U_\alpha \), and take \( a_\alpha \in A \) with \( \rho(x_\alpha, a_\alpha) < 2 \cdot \rho(x_\alpha, A) \). For each \( \alpha \in A \), choose \( \beta_\alpha \in \Lambda_p \) such that \( e_\alpha(\beta_\alpha) = \beta_\alpha(t) = g(a_\alpha) \). Then with each \( x \in Z - A \) there is associated a finite set \( \{ \beta_\alpha | \phi_\alpha(x) > 0 \} \) of elements of \( \Lambda_p \). The extension \( \hat{g}: Z \to \mu_p^{-1}(t) \) is defined in the following steps:

1. For \( x \in Z - A \), let \( \alpha_1 < \alpha_2 < \cdots < \alpha_n \) be the ordering in \( A \) of those elements \( \alpha \) for which \( \phi_\alpha(x) > 0 \), and define
   \[
   \tau(x, \alpha_i) = \frac{\phi_{\alpha_i}(x)}{\phi_{\alpha_1}(x) + \cdots + \phi_{\alpha_n}(x)}, \quad i = 1, 2, \ldots, n.
   \]
   Note that \( \tau(x, \alpha_n) = 1 \).

2. With \( \beta_{\alpha_1}, \ldots, \beta_{\alpha_n} \) the elements of \( \Lambda_p \) corresponding to \( x \), define \( \beta_x = (\beta_{\alpha_1}, \tau(x, \alpha_1); \ldots; \beta_{\alpha_n-1}, \tau(x, \alpha_n); \beta_{\alpha_n}) \).

3. Define \( \hat{g}(x) = e_\alpha(\beta_x) = \beta_x(t) \).

CLAIM 1. \( \hat{g} \) is continuous on \( Z - A \).

Consider \( x \in Z - A \), with \( \alpha_1 < \cdots < \alpha_n \) as in step (1) above. For any \( y \in Z \) sufficiently close to \( x \), each \( \phi_\alpha(y) \) will be near \( \phi_\alpha(x) \), \( i = 1, \ldots, n \). Clearly, this implies that if \( \gamma_1 < \cdots < \gamma_{n+m} \) is the ordering in \( A \) of \( \{ \alpha | \phi_\alpha(y) > 0 \} \), then for each \( k \) such that \( \gamma_k = \alpha_i \) for some \( i \), \( \tau(y, \gamma_k) \) is near \( \tau(x, \alpha_i) \), and for all other \( \gamma_k \), either \( \tau(y, \gamma_k) \) is near 0 or \( \gamma_k < \gamma_k \). It follows that \( \beta_y \) is near \( \beta_x \), and \( \hat{g}(y) \) is near \( \hat{g}(x) \).

CLAIM 2. \( \hat{g} \) is continuous on \( \text{bd} A \).

Consider a \( \alpha \in \text{bd} A \), and let \( y \in Z - A \) denote a point near \( \alpha \). Let \( \{ \alpha_1, \ldots, \alpha_n \} = \{ \alpha \in A | \phi_\alpha(y) > 0 \} \). Then for each \( i \leq n \), the point \( a_\alpha \in A \) associated with \( \alpha_i \) is near \( \alpha \), thus \( \beta_\alpha(t) = g(a_\alpha) \) is near \( g(\alpha) \). Note that the construction of the order arc \( \beta_\alpha(t) \) forces \( \beta_\alpha(t) \subset \beta_{\alpha_1}(t) \cup \beta_{\alpha_2}(t) \cup \cdots \cup \beta_{\alpha_n}(t) \). Since \( M = \bigcup_1^{n+1} \beta_{\alpha_i}(t) \) is an element of \( C_p(X) \) near \( g(\alpha) \in \mu_p^{-1}(t) \), \( \mu(M) \) is near \( t = \mu(\beta_\alpha(t)) \). Since \( \beta_\alpha(t) \subset M \), the nature of the Whitney map \( \mu \) forces \( \beta_\alpha(t) \) and \( M \) to be close. Then \( \hat{g}(y) = \beta_\alpha(t) \) is near \( \hat{g}(\alpha) = g(\alpha) \).

Thus, the extension \( \hat{g}: Z \to \mu_p^{-1}(t) \) of \( g \) is continuous, and this concludes the proof of our main result:

**Theorem.** Each Whitney level \( \mu_p^{-1}(t) \) in \( C_p(X) \) is an AR.

There are several easy corollaries.

**Corollary 1.** \( \{ X \} \) is an unstable point in \( C_p(X) \).

**Proof.** Given \( \varepsilon > 0 \), choose \( t < 1 \) sufficiently close to 1 so that \( \text{diam}_H(\mu_p^{-1}([t, 1])) < \varepsilon \).
Since $\mu_p^{-1}(t)$ is an AR, there is a retraction $r: \mu_p^{-1}([t, 1]) \to \mu_p^{-1}(t)$. Then $r$ extends by the identity to a retraction $R: C_p(X) \to \mu_p^{-1}([0, t])$. We have $R(C_p(X)) \subset C_p(X) - \{X\}$ and $H(R(M), M) < \varepsilon$ for each $M \in C_p(X)$. Thus, $\{X\}$ is unstable in $C_p(X)$.

Note that the above retraction $R$ shows that $\mu_p^{-1}([0, t])$ is an AR. In fact, a similar argument shows that $\mu_p^{-1}([s, t])$ is an AR for all $s < t$.

**Corollary 2.** $C_p(X) - \{X\}$ is an AR.

**Proof.** Since $C_p(X) - \{X\}$ is an ANR, it suffices to show that it is $n$-connected for all $n$ [5]. Let $f: S^n \to C_p(X) - \{X\}$ be a map of the $n$-sphere. Choose $t < 1$ such that $f(S^n) \subset \mu_p^{-1}([0, t])$. Then $f$ is null-homotopic in the AR $\mu_p^{-1}([0, t])$, thus $C_p(X) - \{X\}$ is $n$-connected for all $n$.

If $X$ has a cut point $p$, then clearly $\mu^{-1}(t) = \mu_p^{-1}(t)$ for all $t$ in some neighborhood of 1. Thus we have the following

**Corollary 3.** If $X$ has a cut point, then

(i) for all $t$ in some neighborhood of 1, $\mu^{-1}(t)$ is an AR; and
(ii) $\{X\}$ is unstable in $C(X)$.

The general question suggested by part (ii) has been answered in [1]: For $X$ a Peano continuum, $\{X\}$ is stable in $C(X)$ if and only if $X$ is a finite graph with no cut points.

**BIBLIOGRAPHY**


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