NOTE ON NILPOTENT DERIVATIONS

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Abstract. Let $R$ be a prime ring with center $Z$. Suppose that $d$ is a derivation on $R$ such that $d^n(x) \in Z$ for all $x$, where $n$ is a fixed integer. It is shown that either $d^n(x) = 0$ for all $x \in R$ or $R$ is a commutative integral domain. Moreover, the same conclusion holds even if we assume that $d^n(x) \in Z$ merely for all $x \in I$, where $I$ is a nonzero ideal of $R$.

Let $R$ be a prime ring with center $Z$ and a derivation $d$. Suppose that $d(x) \in Z$ for all $x \in R$, then one can easily show that either $d = 0$ or $R$ is commutative [3, Proof, Theorem 2]. If $d^2(x) \in Z$ for all $x \in R$, we have the same conclusion when $R$ is not of characteristic 2 [3, Theorem 3]. On the other hand, in case char $R = 2$ there exist nonzero derivations $d$ on $R$ such that $d^2 = 0$ provided $R$ is not a domain. One might naturally ask what we can conclude in general if $d^n(x) \in Z$ for all $x \in R$, where $n$ is some fixed integer. Certainly, this condition is fulfilled when $d$ is nilpotent of nilpotency $n$ or when $R$ is a commutative ring. We prove here that these are indeed the only two possibilities where $d^n(R) \subseteq Z$ can happen.

Theorem 1. Let $R$ be a prime ring with center $Z$ and a derivation $d$ on $R$ such that $d^n(R) \subseteq Z$ for some natural number $n$. Then either $d^n = 0$ or $R$ is commutative.

Proof. First note that if $Z = 0$ there is nothing to prove. So we assume that $Z \neq 0$ and proceed to prove by induction on $n$. When $n = 1$, the conclusion holds as we mentioned at the beginning. Now assume that $n > 1$ and the theorem is true for $< n$.

Suppose that char $R = p$; then $\delta = d^p$ is also a derivation on $R$. If $n$ is divisible by $p$, the assumption reads $\delta^{n/p}(R) \subseteq Z$. Since $n/p < n$, it follows from induction hypothesis that either $d^n = \delta^{n/p} = 0$ or $R$ is commutative. So we may henceforth assume that $R$ is of characteristic 0 or a prime $p$ not dividing $n$.

For $\alpha \in Z$ and $x \in R$, we have $d^n(ad^{-1}(x)) \in Z$. That is,

$$d^n(\alpha)d^{-1}(x) + \sum_{i=1}^{n} \binom{n}{i} d^{n-i}(\alpha)d^{i-1}(x) \in Z$$

for all $\alpha \in Z$, $x \in R$. Note that each term in the summation is already in $Z$, therefore $d^n(Z)d^{-1}(R) \subseteq Z$. Consequently, either $d^n(Z) = 0$ or $d^{n-1}(R) \subseteq Z$. If $d^{n-1}(R) \subseteq Z$ we are done by induction hypothesis. So we assume that $d^n(Z) = 0$.

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Let \( m \) be the smallest integer such that \( d^m(Z) = 0 \). Assume first that \( m \geq 2 \). For \( \alpha \in Z \) and \( x \in R \) we have
\[
d^n(d^{m-2}(\alpha)x) = d^{m-2}(\alpha)d^n(x) + nd^{m-1}(\alpha)d^{n-1}(x) \in Z.
\]
Since \( d^{m-2}(\alpha)d^n(x) \in Z \), we have \( nd^{m-1}(\alpha)d^{n-1}(x) \in Z \) for all \( \alpha \in Z \), \( x \in R \). Recall that char \( R \) is 0 or \( p \neq n \), so \( d^{m-1}(Z)d^{n-1}(R) \subseteq Z \) follows. By the minimality of \( m \), \( d^{m-1}(Z) \neq 0 \) so \( d^{n-1}(R) \subseteq Z \), and we are done. Hence it remains to check the case when \( m = 1 \), that is, \( d(Z) = 0 \).

Since \( Z \neq 0 \), we may localize \( R \) at \( Z \) to get a ring \( S = \{a/\alpha | a \in R, \alpha \in Z, \alpha \neq 0\} \). Then \( S \) is a prime ring with center the quotient field of \( Z \). Moreover, we can extend \( d \) to a derivation \( \tilde{d} \) on \( S \) by defining \( \tilde{d}(a/\alpha) = d(a)/\alpha \). Then \( \tilde{d}^n(S) \subseteq Z(S) \), the center of \( S \). So we may replace \( R \) and \( d \) by \( S \) and \( \tilde{d} \), respectively, and assume that \( Z \) is a field. Thus \( d^n(R) \) is a \( Z \)-subspace of dimension at most 1.

Let \( k \) be the smallest integer such that \( d^k(R) \) is finite dimensional over \( Z \). Assume that \( k \geq 1 \). For \( x, y \in R \) we have
\[
d^n(d^{k-1}(x)y) = d^{k-1}(x)d^n(y) + \sum_{i=1}^{n} \binom{n}{i} d^{k+i-1}(x)d^{n-i}(y) \in Z.
\]
Note that \( d^{k+i-1}(R) \) is finite dimensional over \( Z \), so is \( d^{k+i-1}(R)d^{n-i}(y) \) for all \( y \in R \) and \( i \geq 1 \). Thus it follows that \( d^{k-1}(R)d^n(y) \) must be finite dimensional over \( Z \) for each \( y \in R \). Since \( d^n(y) \in Z \), we would have \( d^{k-1}(R) \) finite dimensional to contradict the choice of \( k \) provided \( d^n(y) \) were not zero. Hence we must have \( d^n(R) = 0 \).

At last, assume that \( k = 0 \), that is, \( R \) is finite dimensional over \( Z \). Being a prime ring, \( R \) must be simple. Now \( d(Z) = 0 \), so \( d \) must be inner by a classic result [2, Proposition, p. 100]. In other words, there exists \( a \in R \) such that \( d(x) = ax - xa \) for all \( x \in R \). Thus \( ad^n(x) = d^n(ax) \in Z \), and so either \( a \in Z \) or \( d^n(x) = 0 \) for all \( x \in R \). In either case, we have always \( d^n = 0 \). This completes the proof.

Next we extend the previous theorem by conditioning \( d^n(x) \in Z \) merely for all \( x \) in some nonzero ideal \( I \) of \( R \).

**Theorem 2.** Let \( R \) be a prime ring with center \( Z \) and \( I \) a nonzero ideal of \( R \). Suppose that \( d \) is a derivation on \( R \) such that \( d^n(I) \subseteq Z \) for some natural number \( n \). Then either \( d^n = 0 \) or \( R \) is commutative.

**Proof.** Let \( J = I + d(I) + d^2(I) + \cdots \). Then \( J \) is a nonzero ideal of \( R \), \( d(J) \subseteq J \), and \( d^n(J) \subseteq Z \). Applying Theorem 1, we can conclude that either \( d^n(J) = 0 \) or \( J \) is a commutative ring. However, if \( d^n(J) = 0 \), then \( d^n = 0 \) by a theorem due to Chung and Luh [1]; while if \( J \) is commutative, so is the whole ring \( R \). Thus the theorem is proved.

**References**


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