NOTE ON NILPOTENT DERIVATIONS

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Abstract. Let \( R \) be a prime ring with center \( Z \). Suppose that \( d \) is a derivation on \( R \) such that \( d^n(x) \in Z \) for all \( x \), where \( n \) is a fixed integer. It is shown that either \( d^n(x) = 0 \) for all \( x \in R \) or \( R \) is a commutative integral domain. Moreover, the same conclusion holds even if we assume that \( d^n(x) \in Z \) merely for all \( x \in I \), where \( I \) is a nonzero ideal of \( R \).

Let \( R \) be a prime ring with center \( Z \) and a derivation \( d \). Suppose that \( d(x) \in Z \) for all \( x \in R \), then one can easily show that either \( d = 0 \) or \( R \) is commutative [3, Proof, Theorem 2]. If \( d^2(x) \in Z \) for all \( x \in R \), we have the same conclusion when \( R \) is not of characteristic 2 [3, Theorem 3]. On the other hand, in case \( \text{char } R = 2 \) there exist nonzero derivations \( d \) on \( R \) such that \( d^2 = 0 \) provided \( R \) is not a domain. One might naturally ask what we can conclude in general if \( d^n(x) \in Z \) for all \( x \in R \), where \( n \) is some fixed integer. Certainly, this condition is fulfilled when \( d \) is nilpotent of nilpotency \( n \) or when \( R \) is a commutative ring. We prove here that these are indeed the only two possibilities where \( d^n(R) \subseteq Z \) can happen.

Theorem 1. Let \( R \) be a prime ring with center \( Z \) and \( d \) a derivation on \( R \) such that \( d^n(R) \subseteq Z \) for some natural number \( n \). Then either \( d^n = 0 \) or \( R \) is commutative.

Proof. First note that if \( Z = 0 \) there is nothing to prove. So we assume that \( Z \neq 0 \) and proceed to prove by induction on \( n \). When \( n = 1 \), the conclusion holds as we mentioned at the beginning. Now assume that \( n > 1 \) and the theorem is true for \( < n \).

Suppose that \( \text{char } R = p \); then \( \delta = d^p \) is also a derivation on \( R \). If \( n \) is divisible by \( p \), the assumption reads \( \delta^n/p(R) \subseteq Z \). Since \( n/p < n \), it follows from induction hypothesis that either \( d^n = \delta^n/p = 0 \) or \( R \) is commutative. So we may henceforth assume that \( R \) is of characteristic 0 or a prime \( p \) not dividing \( n \).

For \( \alpha \in Z \) and \( x \in R \), we have \( d^n(ad^{-1}(x)) \in Z \). That is,

\[
d^n(\alpha)d^{-1}(x) + \sum_{i=1}^{n} \binom{n}{i} d^{n-i}(\alpha)d^{n+i-1}(x) \in Z
\]

for all \( \alpha \in Z \), \( x \in R \). Note that each term in the summation is already in \( Z \), therefore \( d^n(Z)d^{-1}(R) \subseteq Z \). Consequently, either \( d^n(Z) = 0 \) or \( d^{n-1}(R) \subseteq Z \). If \( d^{n-1}(R) \subseteq Z \) we are done by induction hypothesis. So we assume that \( d^n(Z) = 0 \).
Let \( m \) be the smallest integer such that \( d^m(Z) = 0 \). Assume first that \( m \geq 2 \). For \( \alpha \in Z \) and \( x \in R \) we have
\[
d^n(d^{m-2}(\alpha)x) = d^{m-2}(\alpha)d^n(x) + nd^{m-1}(\alpha)d^{n-1}(x) \in Z.
\]
Since \( d^{m-2}(\alpha)d^n(x) \in Z \), we have \( nd^{m-1}(\alpha)d^{n-1}(x) \in Z \) for all \( \alpha \in Z, x \in R \). Recall that \( \text{char} R \) is 0 or \( p \nmid n \), so \( d^{m-1}(Z)d^{n-1}(R) \subseteq Z \) follows. By the minimality of \( m \), \( d^{m-1}(Z) \neq 0 \) so \( d^{n-1}(R) \subseteq Z \), and we are done. Hence it remains to check the case when \( m = 1 \), that is, \( d(Z) = 0 \).

Since \( Z \neq 0 \), we may localize \( R \) at \( Z \) to get a ring \( S = \{a/\alpha| a \in R, \alpha \in Z, \alpha \neq 0\} \). Then \( S \) is a prime ring with center the quotient field of \( Z \). Moreover, we can extend \( d \) to a derivation \( \tilde{d} \) on \( S \) by defining \( \tilde{d}(a/\alpha) = d(a)/\alpha \). Then \( \tilde{d}^n(S) \subseteq Z(S) \), the center of \( S \). So we may replace \( R \) and \( d \) by \( S \) and \( \tilde{d} \), respectively, and assume that \( Z \) is a field. Thus \( d^n(R) \) is a \( Z \)-subspace of dimension at most 1.

Let \( k \) be the smallest integer such that \( d^k(R) \) is finite dimensional over \( Z \). Assume that \( k \geq 1 \). For \( x, y \in R \) we have
\[
d^n(d^{k-1}(x)y) = d^{k-1}(x)d^n(y) + \sum_{i=1}^{n} \binom{n}{i} d^k y d^{n-i}(y) \in Z.
\]
Note that \( d^{k+i-1}(R) \) is finite dimensional over \( Z \), so is \( d^{k+i-1}(R)d^{n-i}(y) \) for all \( y \in R \) and \( i \geq 1 \). Thus it follows that \( d^{k-1}(R)d^n(y) \) must be finite dimensional over \( Z \) for each \( y \in R \). Since \( d^n(y) \in Z \), we would have \( d^{k-1}(R) \) finite dimensional to contradict the choice of \( k \) provided \( d^n(y) \) were not zero. Hence we must have \( d^n(R) = 0 \).

At last, assume that \( k = 0 \), that is, \( R \) is finite dimensional over \( Z \). Being a prime ring, \( R \) must be simple. Now \( d(Z) = 0 \), so \( d \) must be inner by a classic result [2, Proposition, p. 100]. In other words, there exists \( a \in R \) such that \( d(x) = ax - xa \) for all \( x \in R \). Thus \( ad^n(x) = d^n(ax) \in Z \), and so either \( a \in Z \) or \( d^n(x) = 0 \) for all \( x \in R \). In either case, we have always \( d^n = 0 \). This completes the proof.

Next we extend the previous theorem by conditioning \( d^n(x) \in Z \) merely for all \( x \) in some nonzero ideal \( I \) of \( R \).

**Theorem 2.** Let \( R \) be a prime ring with center \( Z \) and \( I \) a nonzero ideal of \( R \). Suppose that \( d \) is a derivation on \( R \) such that \( d^n(I) \subseteq Z \) for some natural number \( n \). Then either \( d^n = 0 \) or \( R \) is commutative.

**Proof.** Let \( J = I + d(I) + d^2(I) + \cdots \). Then \( J \) is a nonzero ideal of \( R \), \( d(J) \subseteq J \), and \( d^n(J) \subseteq Z \). Applying Theorem 1, we can conclude that either \( d^n(J) = 0 \) or \( J \) is a commutative ring. However, if \( d^n(J) = 0 \), then \( d^n = 0 \) by a theorem due to Chung and Luh [1]; while if \( J \) is commutative, so is the whole ring \( R \). Thus the theorem is proved.

**References**