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GROWTH OF HARMONIC CONJUGATES IN THE UNIT DISC

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ABSTRACT. Assuming some mild regularity conditions on a positive nondecreasing function $\psi(x) = O(x^a)$ (for some $a > 0$, $x \to \infty$), we show that

$$M_p(r,u) = O\left(\psi\left(\frac{1}{1-r}\right)\right) \quad (r \to 1,\ 0 < p < 1)$$

implies $M_p(r,v) = O(\tilde{\psi}^p(1/(1-r)))^{1/p}$, where $u(z) + iv(z)$ is holomorphic in the open unit disc and

$$\tilde{\psi}^p(x) = \int_{1/2}^{x} \frac{\psi^p(t)}{t}\,dt, \quad x \geqslant \frac{1}{2}.$$

1. Introduction. Throughout this note ψ will denote a positive nondecreasing function defined for real $x \geqslant 0$. For each such function ψ we define another function by

$$\tilde{\psi}(x) = \int_{1/2}^{x} \frac{\psi(t)}{t}\,dt, \qquad x \geqslant \frac{1}{2}.$$

Throughout this paper C denotes a positive constant, not necessarily the same at each occurrence.

A function φ is almost increasing for $x \geqslant 0$ if there exists a positive constant c such that $x_1 < x_2$ implies $\varphi(x_1) \leqslant c\varphi(x_2)$. An almost decreasing function is defined similarly.

Let u be a harmonic function in the open unit disc U and, as usual, denote

$$M_p(r,u) = \left(\frac{1}{2\pi}\int_0^{2\pi} |u(re^{it})|^p\,dt\right)^{1/p}, \qquad 0 < p < \infty,$$

and

$$M_\infty(r,u) = \sup\{|u(re^{it})|, 0 \leqslant t \leqslant 2\pi\}.$$

Assuming that $\psi(x)/x^a$ is almost decreasing for some $a > 0$, A. Shields and D. Williams in [4] showed that if $M_\infty(r,u) = O(\psi(1/(1-r)))$, $r \to 1$, then its conjugate v satisfies $M_\infty(r,v) = O(\tilde{\psi}(1/(1-r)))$, $r \to 1$. They also showed that this

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theorem remains valid if we replace $M_\infty(r, u)$ by $M_1(r, u)$. If $1 < p < \infty$, the well-known theorem of M. Riesz [**1**, p. 54] says that $M_p(r, u) = O(\psi(1/(1-r)))$ implies $M_p(r, v) = O(\psi(1/(1-r)))$.

In this paper we shall be concerned only with means $M_p(r, u)$ when $0 < p < 1$. Hardy and Littlewood [**3**] proved that if $M_p(r, u) = O(1)$ for some $0 < p < 1$, then its conjugate v satisfies $M_p(r, v) = O((\log 1/(1-r))^{1/p})$. They also showed that if $M_p(r, u) = O((1-r)^{-\alpha})$, $\alpha > 0$, $0 < p < 1$, then $M_p(r, v)$ satisfies the same growth condition. We fill the gap between these two results.

THEOREM. *Let u be harmonic in the unit disc. If there exists $a > 0$ such that $\psi(x)/x^a$ is almost decreasing for $x \geqslant 1/2$ and if $M_p(r, u) = O(\psi(1/(1-r)))$, for some p, $0 < p < 1$, then the harmonic conjugate v satisfies*

$$M_p(r, v) = O\left(\left(\tilde{\psi}^p(1/(1-r))\right)^{1/p}\right).$$

If $\psi(x)$ grows like x^α, $\alpha > 0$, then so does $(\tilde{\psi}^p)^{1/p}$ and one obtains the theorem of Hardy and Littlewood. If $\psi(x) \equiv 1$, then $(\tilde{\psi}^p)^{1/p}$ grows like $(\log x)^{1/p}$, thus we recapture the bounded case mentioned above.

2. Proof of the theorem. We will need a lemma.

LEMMA. *Let ψ satisfy the conditions of the theorem. If $0 < p < 1$, then there exists $C > 0$ such that, for all $x \geqslant 1$,*

$$(\tilde{\psi}(x))^p \leqslant C\tilde{\psi}^p(x).$$

PROOF. Since ψ is nondecreasing we have

$$\tilde{\psi}(x) = \int_{1/2}^{x} \frac{\psi(t)}{t}\,dt \leqslant (\psi(x))^{1-p}\tilde{\psi}^p(x).$$

By Lemma 1(ii) of [**4**], $\tilde{\psi}$ grows faster then ψ; there exists $C > 0$ such that, for all $x \geqslant 1$, $\psi(x) \leqslant C\tilde{\psi}(x)$. Hence,

$$(\tilde{\psi}(x))^p \leqslant C\tilde{\psi}^p(x).$$

PROOF OF THE THEOREM. Without loss of generality we may suppose that u is real and $u(0) = 0$. Let $f(z) = u(z) + iv(z) = \sum_{n=1}^\infty \hat{f}(n)z^n$ be a holomorphic function on U. The fractional derivative of f of first order is defined as

$$f^{[1]}(z) = \sum_{n=1}^{\infty} (n+1)\hat{f}(n)z^n.$$

Note that

$$f(z) = \int_0^1 f^{[1]}(tz)\,dt.$$

Let $r_n = 1 - 2^{-n}$. Then

$$\begin{aligned}
|f(re^{i\theta})|^p &= |f(z)|^p = \left|\int_0^1 f^{[1]}(tz)\,dt\right|^p \\
&\leqslant \left(\sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} \sup_{0\leqslant t\leqslant \rho} |f^{[1]}(tz)|\,d\rho\right)^p \\
&\leqslant \left(\sum_{n=1}^{\infty} \sup_{0\leqslant t\leqslant r_n} |f^{[1]}(tz)|2^{-n}\right)^p \\
&\leqslant \sum_{n=1}^{\infty} 2^{-np} \sup_{0\leqslant t\leqslant r_n} |f^{[1]}(tz)|^p \\
&\leqslant C\sum_{n=1}^{\infty} \sup_{0\leqslant t\leqslant r_n} |f^{[1]}(tz)|^p \int_{r_n}^{r_{n+1}} (1-\rho)^{p-1}\,d\rho \\
&\leqslant C\sum_{n=1}^{\infty} \int_{r_n}^{r_{n+1}} \sup_{0\leqslant t\leqslant \rho} |f^{[1]}(tz)|^p (1-\rho)^{p-1}\,d\rho \\
&\leqslant C\int_0^1 (1-\rho)^{p-1} \sup_{0\leqslant t\leqslant \rho} |f^{[1]}(tz)|^p\,d\rho.
\end{aligned}$$

If we now integrate on θ and use the Hardy-Littlewood maximal theorem [**1**, p. 12] we obtain

$$M_p(r,f)^p \leqslant C\int_0^1 (1-t)^{p-1} M_p(tr, f^{[1]})^p\,dt. \tag{1}$$

T. Flett [**2**, p. 762] proved that if $0 < p < 1$ and $1/3 \leqslant r < 1$, then

$$M_p(r,f')^p \leqslant C(1-r)^{-p-1}\int_{(3r-1)/2}^{(1+r)/2} M_p(t,u)^p\,dt.$$

Thus,

$$\begin{aligned}
M_p(r,f')^p &\leqslant C(1-r)^{-p-1}\int_{(3r-1)/2}^{(1+r)/2} \psi^p\left(\frac{1}{1-t}\right)dt \qquad (2)\\
&= C(1-r)^{-p-1}\int_{2(1-r)^{-1}/3}^{2(1-r)^{-1}} [\psi(t)]^p \cdot t^{-2}\,dt \\
&= C(1-r)^{-p-1}\int_{2(1-r)^{-1}/3}^{2(1-r)^{-1}} [\psi(t)/t^a]^p t^{ap-2}\,dt \\
&\leqslant C(1-r)^{ap-p-1}\left[\psi\left(\frac{1}{1-r}\right)\right]^p \int_{2(1-r)^{-1}/3}^{2(1-r)^{-1}} t^{ap-2}\,dt \\
&\leqslant C(1-r)^{-p}\psi^p\left(\frac{1}{1-r}\right).
\end{aligned}$$

The inequality

$$M_p(r,f^{[1]}) \leqslant CM_p(r,f') \tag{3}$$

is obvious since $f^{[1]}(z) = f(z) + zf'(z)$.

Combining (1), (2) and (3) we obtain

$$M_p(r,f)^p \leqslant C\int_0^1 (1-t)^{p-1}(1-tr)^{-p}\psi^p\left(\frac{1}{1-tr}\right)dt. \tag{4}$$

By Lemma 1(iii) of [4],

$$\psi^p\left(\frac{1}{1-r}\right) \leqslant C(1-r)\sum_{n=0}^{\infty}\psi^p(n)r^n.$$

Hence, from (4), it follows that

$$\begin{aligned} M_p(r,f)^p &\leqslant C\int_0^1 \left(\frac{1-rt}{1-t}\right)^{1-p}\left(\sum_{n=0}^{\infty}\psi^p(n)r^n t^n\right)dt \\ &\leqslant C\left(\sum_{n=0}^{\infty}\psi^p(n)r^n\left(\int_0^r \left(\frac{1-rt}{1-t}\right)^{1-p} t^n\,dt\right)\right. \\ &\qquad \left.+\sum_{n=0}^{\infty}\psi^p(n)r^n\left(\int_r^1 \left(\frac{1-rt}{1-t}\right)^{1-p} t^n\,dt\right)\right). \end{aligned} \tag{5}$$

We now show that each term in (5) is $O(\tilde{\psi}^p(1/(1-r)))$. Applying Lemma 1(v) of [4] to the function $\psi^p(x)$ we have

$$\sum_{n=0}^{\infty}\frac{\psi^p(n)}{n+1}r^n \leqslant C\tilde{\psi}^p\left(\frac{1}{1-r}\right).$$

Hence,

$$\begin{aligned} \sum_{n=0}^{\infty}\psi^p(n)r^n\left(\int_0^r \left(\frac{1-rt}{1-t}\right)^{1-p} t^n\,dt\right) &\leqslant C\sum_{n=0}^{\infty}\psi^p(n)\left(\int_0^1 t^n\,dt\right)r^n \\ &\leqslant C\sum_{n=0}^{\infty}\frac{\psi^p(n)}{n+1}r^n \leqslant C\tilde{\psi}^p\left(\frac{1}{1-r}\right). \end{aligned}$$

It is a simple consequence of Jensen's inequality that

$$(1-r)\sum_{n=0}^{\infty}\left[\frac{\psi(n)}{n+1}\right]^p r^n \leqslant \left\{(1-r)\cdot\sum_{n=0}^{\infty}\frac{\psi(n)}{n+1}r^n\right\}^p. \tag{6}$$

For the second term, we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\psi^p(n)r^n\left(\int_r^1 \left(\frac{1-rt}{1-t}\right)^{1-p} t^n\,dt\right) \\ &\qquad \leqslant C(1-r)^{1-p}\sum_{n=0}^{\infty}\psi^p(n)\left(\int_0^1 (1-t)^{p-1}t^n\,dt\right)r^n \\ &\qquad \leqslant C(1-r)^{1-p}\sum_{n=0}^{\infty}\psi^p(n)\frac{\Gamma(p)\Gamma(n+1)}{\Gamma(n+p+1)}r^n \\ &\qquad \leqslant C(1-r)^{1-p}\sum_{n=0}^{\infty}\frac{\psi^p(n)}{(n+1)^p}r^n. \end{aligned}$$

From (6), Lemma 1(v) of [**4**], and the lemma it follows that the second term on the right-hand side of (5) is also $O(\tilde{\psi}^p(1/(1-r)))$.

3. Remarks. A function ψ is normal if there exist a, $b > 0$ such that $\psi(x)/x^a$ is almost decreasing for $x \geqslant 1/2$ and $\psi(x)/x^b$ is almost increasing for $x \geqslant 1/2$. If $0 < p < 1$ and ψ is normal, then ψ, $\tilde{\psi}$ and $(\tilde{\psi}^p)^{1/p}$ have the same rate of growth. Thus, when ψ is normal the theorem says that if u is harmonic in the unit disc and $M_p(r,u) = O(\psi(1/(1-r)))$, then $M_p(r,v) = O(\psi(1/(1-r)))$. If ψ satisfies the conditions of the theorem, but is not normal, then $(\tilde{\psi}^p)^{1/p}$ grows at a faster rate than ψ. I do not know whether the theorem is best possible, i.e., is there a harmonic function u such that

$$M_p(r,u) = O\left(\psi\left(\frac{1}{1-r}\right)\right) \quad \text{and} \quad M_p(r,v) \geqslant C\left(\tilde{\psi}^p\left(\frac{1}{1-r}\right)\right)^{1/p}?$$

We note that Hardy and Littlewood proved in [**3**] that if k is a positive integer, $p = 1/(k+1)$ and $f(z) = \exp(\frac{1}{2}k\pi i)(1-z)^{-k-1}$ then $M_p(r, \operatorname{Re} f)$ is bounded and $M_p(r,f) \sim (\log(1/(1-r)))^{1/p}$, $r \to 1$.

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