GROWTH OF HARMONIC CONJUGATES IN THE UNIT DISC

MIROLJUB JEVTIĆ

Abstract. Assuming some mild regularity conditions on a positive nondecreasing function \( \psi(x) = O(x^a) \) (for some \( a > 0 \), \( x \to \infty \)), we show that

\[ M_p(r, u) = O \left( \psi \left( \frac{1}{1 - r} \right) \right) \quad (r \to 1, \ 0 < p < 1) \]

implies \( M_p(r, v) = O(\psi^p(1/(1 - r)))^{1/p} \), where \( u(z) + iv(z) \) is holomorphic in the open unit disc and

\[ \psi^p(x) = \int_{1/2}^{x} \frac{\psi(t)}{t} \, dt, \quad x \geq \frac{1}{2}. \]

1. Introduction. Throughout this note \( \psi \) will denote a positive nondecreasing function defined for real \( x \geq 0 \). For each such function \( \psi \) we define another function by

\[ \psi(x) = \int_{1/2}^{x} \frac{\psi(t)}{t} \, dt, \quad x \geq \frac{1}{2}. \]

Throughout this paper \( C \) denotes a positive constant, not necessarily the same at each occurrence.

A function \( \varphi \) is almost increasing for \( x > 0 \) if there exists a positive constant \( c \) such that \( x_1 < x_2 \) implies \( \varphi(x_1) \leq c\varphi(x_2) \). An almost decreasing function is defined similarly.

Let \( m \) be a harmonic function in the open unit disc \( U \) and, as usual, denote

\[ M_p(r, u) = \left( \frac{1}{2\pi} \int_0^{2\pi} |u(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty, \]

and

\[ M_\infty(r, u) = \sup\{ |u(re^{it})|, 0 \leq t \leq 2\pi \}. \]

Assuming that \( \psi(x)/x^a \) is almost decreasing for some \( a > 0 \), A. Shields and D. Williams in [4] showed that if \( M_\infty(r, u) = O(\psi(1/(1 - r))) \), \( r \to 1 \), then its conjugate \( v \) satisfies \( M_\infty(r, v) = O(\psi^p(1/(1 - r))) \), \( r \to 1 \). They also showed that this
theorem remains valid if we replace $M_{\infty}(r, u)$ by $M_1(r, u)$. If $1 < p < \infty$, the well-known theorem of M. Riesz [1, p. 54] says that $M_p(r, u) = O(\psi(1/(1 - r)))$ implies $M_p(r, v) = O(\psi(1/(1 - r)))$.

In this paper we shall be concerned only with means $M_p(r, u)$ when $0 < p < 1$. Hardy and Littlewood [3] proved that if $M_p(r, u) = O(1)$ for some $0 < p < 1$, then its conjugate $v$ satisfies $M_p(r, v) = O((\log 1/(1 - r))^{1/p})$. They also showed that if $M_p(r, u) = O((1 - r)^{-a})$, $a > 0$, $0 < p < 1$, then $M_p(r, v)$ satisfies the same growth condition. We fill the gap between these two results.

**Theorem.** Let $u$ be harmonic in the unit disc. If there exists $a > 0$ such that $\psi(x)/x^a$ is almost decreasing for $x \geq 1/2$ and if $M_p(r, u) = O(\psi(1/(1 - r)))$, for some $p, 0 < p < 1$, then the harmonic conjugate $v$ satisfies

$$M_p(r, v) = O\left((\psi^p(1/(1 - r)))^{1/p}\right).$$

If $\psi(x)$ grows like $x^a$, $a > 0$, then so does $(\psi^p)^{1/p}$ and one obtains the theorem of Hardy and Littlewood. If $\psi(x) = 1$, then $(\psi^p)^{1/p}$ grows like $(\log x)^{1/p}$, thus we recapture the bounded case mentioned above.

**2. Proof of the theorem.** We will need a lemma.

**Lemma.** Let $\psi$ satisfy the conditions of the theorem. If $0 < p < 1$, then there exists $C > 0$ such that, for all $x \geq 1$,

$$\left(\psi(x)\right)^p \leq C\psi^p(x).$$

**Proof.** Since $\psi$ is nondecreasing we have

$$\psi(x) = \int_{1/2}^x \frac{\psi(t)}{t} \, dt \leq (\psi(x))^{1-p} \psi^p(x).$$

By Lemma 1(ii) of [4], $\psi$ grows faster then $\psi$; there exists $C > 0$ such that, for all $x \geq 1$, $\psi(x) \leq C\psi(x)$. Hence,

$$\left(\psi(x)\right)^p \leq C\psi^p(x).$$

**Proof of the theorem.** Without loss of generality we may suppose that $u$ is real and $u(0) = 0$. Let $f(z) = u(z) + iv(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ be a holomorphic function on $U$. The fractional derivative of $f$ of first order is defined as

$$f^{[1]}(z) = \sum_{n=1}^{\infty} (n + 1) \hat{f}(n) z^n.$$

Note that

$$f(z) = \int_0^1 f^{[1]}(tz) \, dt.$$
Let $r_n = 1 - 2^{-n}$. Then

$$|f(re^{i\theta})|^p = |f(z)|^p = \left| \int_0^1 f^{[1]}(tz) \, dt \right|^p$$

$$\leq \left( \sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} \sup_{0 \leq t \leq \rho} |f^{[1]}(tz)| \, d\rho \right)^p$$

$$\leq \left( \sum_{n=1}^{\infty} \sup_{0 \leq t \leq r_n} |f^{[1]}(tz)| |2^{-n}| \right)^p$$

$$\leq \sum_{n=1}^{\infty} 2^{-np} \sup_{0 \leq t \leq r_n} |f^{[1]}(tz)|^p$$

$$\leq C \sum_{n=1}^{\infty} \sup_{0 \leq t \leq r_n} |f^{[1]}(tz)| \int_{r_n}^{r_{n+1}} (1 - \rho)^{p-1} \, d\rho$$

$$\leq C \sum_{n=1}^{\infty} \int_{r_n}^{r_{n+1}} \sup_{0 \leq t \leq \rho} \left| f^{[1]}(tz) \right|^p (1 - \rho)^{p-1} \, d\rho$$

$$\leq C \int_0^1 (1 - \rho)^{p-1} \sup_{0 \leq t \leq \rho} \left| f^{[1]}(tz) \right|^p \, d\rho.$$

If we now integrate on $\theta$ and use the Hardy-Littlewood maximal theorem [1, p. 12] we obtain

(1) $M_p(r,f)^p \leq C \int_0^1 (1 - t)^{p-1} M_p(tr,f^{[1]})^p \, dt.$

T. Flett [2, p. 762] proved that if $0 < p < 1$ and $1/3 \leq r < 1$, then

$M_p(r,f')^p \leq C (1 - r)^{-p-1} \int_{(3r-1)/2}^{(1+r)/2} M_p(t,u)^p \, dt.$

Thus,

(2) $M_p(r,f')^p \leq C (1 - r)^{-p-1} \int_{(3r-1)/2}^{(1+r)/2} \psi \left( \frac{1}{1-t} \right) dt$

$$= C (1 - r)^{-p-1} \int_{2(1-r)^{-1}}^{2(1-r)^{-1}} \left[ \psi (t) \right]^p \cdot t^{-2} \, dt$$

$$= C (1 - r)^{-p-1} \int_{2(1-r)^{-1}}^{2(1-r)^{-1}} \left[ \psi (t) \right]^p t^{-2} \, dt$$

$$\leq C (1 - r)^{ap^{-p-1}} \left[ \psi \left( \frac{1}{1-r} \right) \right]^p \int_{2(1-r)^{-1}}^{2(1-r)^{-1}} t^{-2} \, dt$$

$$\leq C (1 - r)^{-p} \psi \left( \frac{1}{1-r} \right).$$

The inequality

(3) $M_p(r,f^{[1]}) \leq CM_p(r,f')$

is obvious since $f^{[1]}(z) = f(z) + zf'(z)$. 

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Combining (1), (2) and (3) we obtain

\[(4) \quad M_p(r, f)^p \leq C \int_0^1 (1 - t)^{p-1}(1 - tr)^p \psi_p \left( \frac{1}{1 - tr} \right) dt.\]

By Lemma 1(iii) of [4],

\[\psi_p \left( \frac{1}{1 - r} \right) \leq C(1 - r) \sum_{n=0}^{\infty} \psi_p(n) r^n.\]

Hence, from (4), it follows that

\[(5) \quad M_p(r, f)^p \leq C \int_0^1 \left( \frac{1 - rt}{1 - t} \right)^{1-p} \left( \sum_{n=0}^{\infty} \psi_p(n) r^n t^n \right) dt \]

\[\leq C \left( \sum_{n=0}^{\infty} \psi_p(n) r^n \left( \int_0^1 \left( \frac{1 - rt}{1 - t} \right)^{1-p} t^n dt \right) \right) \]

\[\quad + \sum_{n=0}^{\infty} \psi_p(n) r^n \left( \int_0^1 \left( \frac{1 - rt}{1 - t} \right)^{1-p} t^n dt \right).\]

We now show that each term in (5) is \(O(\psi_p(1/(1 - r)))\). Applying Lemma 1(v) of [4] to the function \(\psi_p(x)\) we have

\[\sum_{n=0}^{\infty} \psi_p(n) \frac{r^n}{n + 1} \leq C \psi_p \left( \frac{1}{1 - r} \right).\]

Hence,

\[\sum_{n=0}^{\infty} \psi_p(n) r^n \left( \int_0^1 \left( \frac{1 - rt}{1 - t} \right)^{1-p} t^n dt \right) \leq C \sum_{n=0}^{\infty} \psi_p(n) \left( \int_0^1 t^n dt \right) r^n \]

\[\leq C \sum_{n=0}^{\infty} \psi_p(n) r^n \leq C \psi_p \left( \frac{1}{1 - r} \right).\]

It is a simple consequence of Jensen's inequality that

\[(6) \quad (1 - r) \sum_{n=0}^{\infty} \left( \frac{\psi(n)}{n + 1} \right)^p r^n \leq \left( (1 - r) \cdot \sum_{n=0}^{\infty} \frac{\psi(n)}{n + 1} r^n \right)^p.\]

For the second term, we have

\[\sum_{n=0}^{\infty} \psi_p(n) r^n \left( \int_r^1 \left( \frac{1 - rt}{1 - t} \right)^{1-p} t^n dt \right) \]

\[\leq C(1 - r)^{1-p} \sum_{n=0}^{\infty} \psi_p(n) \left( \int_0^1 (1 - t)^{p-1} t^n dt \right) r^n \]

\[\leq C(1 - r)^{1-p} \sum_{n=0}^{\infty} \psi_p(n) \frac{\Gamma(p) \Gamma(n + 1)}{\Gamma(n + p + 1)} r^n \]

\[\leq C(1 - r)^{1-p} \sum_{n=0}^{\infty} \frac{\psi_p(n)}{(n + 1)^p} r^n.\]
From (6), Lemma 1(v) of [4], and the lemma it follows that the second term on the right-hand side of (5) is also $O(\tilde{\psi}^{p}(1/(1 - r)))$.

3. Remarks. A function $\psi$ is normal if there exist $a, b > 0$ such that $\psi(x)/x^a$ is almost decreasing for $x \geq 1/2$ and $\psi(x)/x^b$ is almost increasing for $x \geq 1/2$. If $0 < p < 1$ and $\psi$ is normal, then $\psi, \tilde{\psi}$ and $(\tilde{\psi}^{p})^{1/p}$ have the same rate of growth. Thus, when $\psi$ is normal the theorem says that if $u$ is harmonic in the unit disc and $M_p(r, u) = O(\psi(1/(1 - r)))$, then $M_p(r, v) = O(\psi(1/(1 - r)))$. If $\psi$ satisfies the conditions of the theorem, but is not normal, then $(\tilde{\psi}^{p})^{1/p}$ grows at a faster rate than $\psi$. I do not know whether the theorem is best possible, i.e., is there a harmonic function $u$ such that

$$M_p(r, u) = O\left(\psi\left(\frac{1}{1 - r}\right)\right) \quad \text{and} \quad M_p(r, v) \geq C\left(\tilde{\psi}^{p}\left(\frac{1}{1 - r}\right)\right)^{1/p}?$$

We note that Hardy and Littlewood proved in [3] that if $k$ is a positive integer, $p = 1/(k + 1)$ and $f(z) = \exp(\frac{k}{2} k \pi i)(1 - z)^{-k-1}$ then $M_p(r, \text{Re} f)$ is bounded and $M_p(r, f) \sim (\log(1/(1 - r)))^{1/p}, \; r \to 1$.

REFERENCES


INSTITUT ZA MATHEMATIKU, STUDENTSKI TRG 16, 11000 BEograd, Yugoslavia