PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 1, September 1986

UNIQUENESS FOR SINGULAR BACKWARD PARABOLIC INEQUALITIES

ALAN V. LAIR

ABSTRACT. We prove that the only solution of the partial differential inequality

$$\left(u_t + \sum_{i=1}^{n}\left[u_{x_i x_i} + \left(\frac{k_i}{x_i}\right)u_{x_i}\right]\right)^2 \leqslant c\left[u^2 + |u_x|^2\right]$$

on a bounded region with homogeneous initial and boundary conditions is the trivial one.

Introduction. In this paper we consider the differential inequality

$$(L[u])^2 \leqslant c\left[u^2 + |u_x|^2\right], \qquad (x,t) \text{ in } Q, \tag{0.1}$$

where

$$L[u] \equiv u_t + A[u], \quad A[u] \equiv \sum_{i=1}^{n}\left[u_{x_i x_i} + \left(\frac{k_i}{x_i}\right)u_{x_i}\right],$$

$$Q_T = D \times (0,T] \subset R^{n+} \times (0,T], \quad R^{n+} = \{(x_1,\ldots,x_n)\colon x_i > 0, 1 \leqslant i \leqslant n\},$$

c, $k_1,\ldots,k_n$ are arbitrary constants and $|u_x|$ is the euclidean norm of the spatial gradient of u. We assume that the set D is bounded with piecewise smooth boundary B, which in general has a nonempty intersection with each of the faces $x_i = 0$. We show that the only solution of (0.1) with homogeneous boundary and initial conditions is the trivial one. In some instances (which are dictated by the values of the k_i) the homogeneity condition on the boundary may be relaxed somewhat to include only part of the boundary. The boundary conditions may be either of the Dirichlet or Neumann types.

Recently Ames [3] and Young [12] proved uniqueness for the initial-boundary value problem of the equation

$$L[u] + c(t)u = 0 \tag{0.2}$$

in Q_T with the set $D = \{(x_1,\ldots,x_n)\colon 0 < x_i < a_i, 1 \leqslant i \leqslant n\}$. Both proofs work equally well for both Dirichlet and Neumann boundary conditions. However the proofs in these papers depend a great deal on the form of (0.2). In particular, neither result applies to equations of the form

$$L[u] = f(x,t,u,u_x) \tag{0.3}$$

Received by the editors October 15, 1964 and, in revised form, August 5, 1985.
1980 *Mathematics Subject Classification*. Primary 35K20, 35K55; Secondary 35K65.

unless the function f has the particular form $c(t)u$. Our result applies for any function f in (0.3) which is uniformly Lipschitz continuous in both u and u_x.

Young's proof [**12**] involves the use of a method similar to that developed in the study of uniqueness of boundary value problems for the hyperbolic equations. (See e.g. [**11**] and its references.) The method involves the caluclation of a complete set of eigenfunctions related to the operator A, multiplication of equation (0.2) by an arbitrary eigenfunction, integrating over D, and showing that the inner product of the eigenfunction and $u(\cdot, t)$ is zero, thus producing $u = 0$. One of the problems with Young's results is that all of the k_i's are required to have the same sign. Ames [**3**] oversomes this difficulty by using a logarithmic convexity argument [**10**]. However, in order to do this, Ames needs strong requirements placed on the function at the face $x_i = 0$ if $k_i \leqslant -1$ in that uu_{x_i} must approach zero faster than $x_i^{k_i}$. Young [**12**] does not need such a condition since the nature of the eigenfunctions at the face $x_i = 0$ is well known, and thus not as much *a priori* knowledge about the function u is needed at that face.

In the present paper we show that inequality (0.1) has only the trivial solution under basically the same hypothesis as used in Ames [**3**]. However, it should be noted that although some of these conditions are not expressly given in [**3**], it is clear from the proofs given therein that they are required.

Physically, when n is unity and $k = k_1$ is a positive integer, the operator L represents the backward diffusion operator in $k + 1$ space in which radial symmetry is assumed. For this case, several authors have contributed results for the forward-in-time case. Among these are Alexiades [**1, 2**], who studies the well-posedness of the boundary value problem for the forward equation

$$L_0[u] \equiv u_t - A[u] = f(x, t),$$

and Arena [**4**], who considers an alternative form of the homogeneous equation $L_0[u] = 0$ given by $u_t = xu_{xx}$ for $k = -1$ and $c = 0$ and studies the Cauchy problem. Likewise the Fokker-Planck equation [**6**] takes on a form similar to Arena's except that lower order terms are included, and it too, for the backward-in-time case, can be put into the form of (0.3). Applications and derivations of related equations in statistics can be found in [**5**, Chapter X].

1. Main results. Our method of proof involves a method originally reported in [**9**] and used later in [**7**, Chapter 6] and [**8**]. However because of the singular coefficient in the principal part of the operator L, very significant changes in the method must be incorporated into the present problem. One of these changes involves, oddly enough, a time scale change which introduces a time degeneracy into the principal part of the operator L. Let $s^2 = t$, $v(x, s) = u(x, t)$ and note that

$$2su_t(x, t) = v_s(x, s). \tag{1.1}$$

Now multiply (0.1) by s^2, use (1.1) and multiply by 4 to get

$$(L_s[v])^2 \leqslant 4c_0 s^2 v^2 + 4c_1 s^2 |v_x|^2, \tag{1.2}$$

where $L_s[v] \equiv v_s + 2A_s[v]$ and $A_s[v] \equiv sA[v]$. Thus our problem now becomes one of showing that the only solution v of (1.2) on the set Q_S ($S^2 = T$) with homogeneous boundary and initial conditions is the trivial one.

We use the following notation with $p(x) = \prod_{i=1}^n x_i^{k_i}$:

$$(u,w) = \int_{Q_S} u(x,s)w(x,s)p(x)\,dx\,ds,$$

$$\|w\|^2 = (w,w) \quad \text{and} \quad \|w_x\|^2 = \sum_{i=1}^n (w_{x_i}, w_{x_i}).$$

We let P_S be the set of functions w ($= w(x,s)$) in $C^1(\overline{Q}_S) \cap C^2(Q_S)$ such that $pw|w_x|$ and $pw_t|w_x|$ are in $C(\overline{Q}_S)$ (or more rigorously, there exist functions F and G in $C(\overline{Q}_S)$ such that $F = pw|w_x|$ and $G = pw_t|w_x|$ on Q_S) and the norms $\|v\|$, $\|v_s\|$, and $\|v_x\|$ are finite. We let P_S^0 denote the set of functions w in P_S for which

$$w(x,0) = w(x,T) = 0 \quad \text{on } D \tag{1.3}$$

and

$$pww_N = 0 \quad \text{on } B \times [0,S]. \tag{1.4}$$

Here w_N denotes the derivative normal to B. Finally, we let $r(s) = s + b$, where b is an arbitrary positive number.

LEMMA. *If $v \in P_S^0$, then for any positive integer m we have*

$$\|r^{-m}L_s[v]\|^2 \geqslant m\|r^{-m-1}v\|^2 + 2\|r^{-m}v_x\|^2. \tag{1.5}$$

PROOF. Let $z(x,s) = r(s)^{-m}v(x,s)$ and note that the identity

$$r^{-m}L_s[v] = r^{-m}L[r^m z] = z_s + mr^{-1}z + 2A_s[z]$$

may be used to obtain

$$\begin{aligned} \|r^{-m}L_s[v]\|^2 &= \|z_s\|^2 + 4(z_s, A_s[z]) + 2m(z_s, r^{-1}z) + \|mr^{-1}z + 2A_s[z]\|^2 \\ &\geqslant 4(z_s, A_s[z]) + 2m(z_s, r^{-1}z). \end{aligned} \tag{1.6}$$

We now simplify the last two expressions in (1.6). In the first of these, we use the identity $pA_s[z] = \sum_{i=1}^n s(pz_{x_i})_{x_i}$ to obtain

$$4(z_s, A_s[z]) = 4\sum_{i=1}^n \int_{Q_S} sz_s(pz_{x_i})_{x_i}\,dx\,ds. \tag{1.7}$$

We now integrate the right side of (1.7) by parts on x, then on s using (1.3) and (1.4) in the process, to get

$$\begin{aligned} 4(z_s, A_s[z]) &= -4\sum_{i=1}^n \int_{Q_S} sz_{sx_i}z_{x_i}p\,dx\,ds \\ &= -2\int_{Q_S} s\left(|z_x|^2\right)_s p\,dx\,ds = 2\|z_x\|^2 = 2\|r^{-m}v_x\|^2. \end{aligned} \tag{1.8}$$

Notice that for the above integration by parts to be valid, we need $pz_sz_N = 0$ on $B \times [0,S]$, which is indeed true for all $z \in P_S^0$. Similarly, in the last expression in

(1.6), we may integrate by parts with respect to s and use (1.3) to get

$$(1.9)\qquad 2m(z_s, r^{-1}z) = m\int_{Q_S} r^{-1}(z^2)_s p\,dx\,dx$$

$$= m\int_{Q_S} r^{-2}z^2 p\,dx\,ds = m\|r^{-m-1}v\|^2.$$

Substituting (1.8) and (1.9) into (1.6) yields (1.5). This completes the proof.

THEOREM. *Suppose $v \in P_S$ and satisfies* (1.2) *as well as the conditions*

$$(1.10)\qquad v(x,0) = 0 \quad \text{on } D,$$

$$(1.11)\qquad pvv_N = 0 \quad \text{on } B \times [0,S].$$

Then $v = 0$ on Q_S.

REMARK. It is interesting to note that, for each $k_i > 0$, (1.11) implies that no condition on v (or v_N) is required on the part of the boundary B which intersects the fact $x_i = 0$. On the other hand, if $k_i \leqslant -1$, a strong boundary condition on v (or v_N) is needed at that face. In the event that $-1 < k_i \leqslant 0$, either homogeneous Dirichlet or Neumann boundary conditions at the face $B \cap \{x_i = 0\}$ will ensure that (1.11) holds.

PROOF OF THE THEOREM. To prove the desired result, it suffices to show that $v(x,s) = 0$ for all $(x,s) \in Q_S$ for some positive S depending only on the constant c from (0.1), for then analysis very similar to that below may be used to show that v vanishes on $[S, 2S]$, $[2S, 3S], \ldots$ to obtain $v = 0$ on Q_S for any $S > 0$. In particular, we shall need $S^2 \leqslant 2/c$ provided $c > 0$. If $c = 0$, then the analysis works for arbitrary finite S.

Now let s_1 be an arbitrary number in $(0, S)$. We shall show that $v = 0$ on $D \times [0, s_1]$. For this purpose choose s_2 and s_3 such that $0 < s_1 < s_2 < s_3 < S$. Let g be a real-valued, infinitely differentiable function defined on $[0, S]$ such that $g(s) = 1$ for $0 \leqslant s \leqslant s_2$, $g(s) = 0$ for $s_3 \leqslant s \leqslant S$, and $0 \leqslant g(s) \leqslant 1$ for $s_2 \leqslant s \leqslant s_3$. Now set $w(x,s) = g(s)v(x,s)$ and observe that $w \in P_S^0$. Using the lemma and the fact that $w = v$ on $[0, s_2]$, we may use estimates similar to those of inequality (2.16) of [**8**] to get

$$(1.12)\qquad I_m \equiv (s_2 + b)^{-2m}\int_{s_2}^{s_3}\int_D (L_s[w])^2 p\,dx\,ds$$

$$\geqslant \int_0^{s_2}\int_D (m - cb^2s^2)r^{-2m-2}v^2 p\,dx\,ds$$

$$+ \int_0^{s_2}\int_D (2 - cs^2)r^{-2m}|v_x|^2 p\,dx\,ds.$$

Since $s_2^2 < S^2 \leqslant 2/c$, the second integral on the right side of (1.12) is nonnegative. Similarly, using the estimates $m - cb^2s^2 \geqslant m - 2b^2$ and $r^{-2m-2} \geqslant (s_1 + b)^{-2m-2}$

for $0 \leqslant s \leqslant s_1$ in the first integral on the right side of (1.12) gives

$$I_m \geqslant (m - 2b^2)\int_0^{s_2}\int_D r^{-2m-2}v^2 p\,dx\,ds \tag{1.13}$$

$$\geqslant (m - 2b^2)(s_1 + b)^{-2m-2}\int_0^{s_2}\int_D v^2 p\,dx\,ds$$

for $m > 2b^2$. Rearranging the terms in (1.13) now yields

$$\begin{aligned}(s_1 + b)^{2m+2}(s_2 + b)^{-2m}(m - 2b^2)^{-1}\int_{s_2}^{s_3}\int_D (L_s[w])^2 p\,dx\,ds \\ \geqslant \int_0^{s_1}\int_D v^2 p\,dx\,ds.\end{aligned} \tag{1.14}$$

Letting m approach infinity makes the left side of (1.14) approach zero. Since the right side is independent of m, it must be zero. Thus $v = 0$ on $D \times [0, s_1]$, and since s_1 was arbitrarily chosen on $[0, S)$, we have $v = 0$ on $D \times [0, S]$. This completes the proof.

References

1. V. Alexiades, *Generalized axially symmetric heat potentials and singular parabolic initial-boundary value problems*, Arch. Rational Mech. Anal. **79** (1982), 325–350.

2. _____, *A singular parabolic initial-boundary value problem in non-cylindrical domain*, SIAM J. Math. Anal. **11** (1980), 348–357.

3. K. A. Ames, *Uniqueness and continuous dependence results for solutions of singular differential equations*, J. Math. Anal. Appl. **103** (1984), 172–183.

4. O. Arena, *On a degenerate elliptic-parabolic equation*, Comm. Partial Differential Equations **3** (1978), 1007–1040.

5. W. Feller, *An introduction to probability theory and its applications*. Vol. II, Wiley, New York, 1966.

6. _____, *Two singular diffusion problems*, Ann. of Math. (2) **54** (1951), 173–182.

7. A Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

8. A. V. Lair, *A unique continuation theorem involving a degenerate parabolic operator*, Proc. Amer. Math. Soc. **66** (1977), 41–45.

9. M. Lees and M. H. Protter, *Unique continuation for parabolic equations and inequalities*, Duke Math. J. **28** (1961), 369–382.

10. L. E. Payne, *Improperly posed problems in partial differential equations*, SIAM, Philadelphia, Pa., 1975.

11. C. C. Travis and E. C. Young, *Uniqueness of solutions to singular boundary value problems*, SIAM J. Math. Anal. **8** (1977), 111–117.

12. E. C. Young, *Uniqueness of solution of a singular heat equation*, Internat. J. Math. Math. Sci. **7** (1984), 201–204.

Department of Mathematics and Computer Science (AFIT / ENC), Air Force Institute of Technology, Wright - Patterson AFB, Ohio 45433