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UNIFORM ERGODIC THEOREMS FOR LOCALLY INTEGRABLE SEMIGROUPS AND PSEUDO-RESOLVENTS

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Dedicated to Professor Ky Fan

ABSTRACT. We study uniform ergodicity (at ∞) of a locally integrable operator semigroup $T(\cdot)$ of type w_0 under a suitable condition which is weaker than the usual one "$w_0 \leqslant 0$". We also give a precise characterization of the uniform Cesàro-ergodicity for semigroups of class $(0, A)$. To prove the part of Abel-ergodicity we first prove a general uniform ergodic theorem for pseudo-resolvents.

1. Introduction. Let $B(X)$ be the Banach algebra of all bounded linear operators on a Banach space X, and let $\{T(t); t > 0\}$ be a family in $B(X)$ with the properties: (i) $T(s+t) = T(s)T(t)$ for all $s, t > 0$; (ii) for each $x \in X$ the function $T(\cdot)x$ is Bochner integrable with respect to the Lebesgue measure over every finite subinterval of $(0, \infty)$. Such a family $T(\cdot)$ is called a locally integrable semigroup. It is known to be strongly continuous on $(0, \infty)$.

The type of $T(\cdot)$ is the number $w_0 := \inf_{t>0} t^{-1} \log\|T(t)\| < \infty$. For every λ with $\operatorname{Re}\lambda > w_0$ and $x \in X$ the Bochner integral

$$R(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)x \, dt$$

exists and defines a bounded linear operator $R(\lambda)$ on X. The function $R(\lambda)$, $\operatorname{Re}\lambda > w_0$ (called the Laplace transform of $T(\cdot)$), satisfies the first resolvent equation

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu),$$

that is, $R(\cdot)$ is a pseudo-resolvent on $\{\lambda \in \mathbf{C}; \operatorname{Re}\lambda > w_0\}$ (cf. [**2**, p. 510]). $R(\cdot)$ has a unique maximal extension satisfying the above equation, and the domain of definition Ω of this maximally extended pseudo-resolvent, which we shall still denote by $R(\cdot)$, is an open subset of the complex plane $\mathbf{C}$, on which $R(\cdot)$ is analytic and cannot be continued analytically beyond (cf. [**2**, pp. 188–189]). We denote $w := \inf\{u \in (-\infty, \infty); \lambda \in \Omega$ for all λ with $\operatorname{Re}\lambda > u\}$. Then $w \leqslant w_0$.

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One can also define the Laplace transform $R_s(\cdot)$ of $T(\cdot)$ in the following weaker sense:

$$R_s(\lambda) := \text{so-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)\, ds,$$

where the integral represents the operator which maps each $x \in X$ to the corresponding Bochner integral $\int_0^t e^{-\lambda s} T(s) x\, ds$. It can be proved (cf. [**1**, p. 248]) that if for a number λ_0 $R_s(\lambda_0)$ exists, then $R_s(\lambda)$ exists for all λ with $\operatorname{Re}\lambda > \operatorname{Re}\lambda_0$. Let us denote

$$\begin{aligned}\sigma &:= \inf\{u \in (-\infty, \infty);\ R_s(\lambda) \text{ exists for all } \lambda \text{ with } \operatorname{Re}\lambda > u\} \\ &= \inf\{u \in (-\infty, \infty);\ R_s(u) \text{ exists}\}, \\ \sigma_a &:= \inf\{u \in (-\infty, \infty);\ R_s(u) \text{ is analytic for all } \lambda \text{ with } \operatorname{Re}\lambda > u\}.\end{aligned}$$

It is clear that $\sigma \leqslant \sigma_a$, $w \leqslant \sigma_a \leqslant w_0$, and that $R_s(\lambda) = R(\lambda)$ for all λ with $\operatorname{Re}\lambda > w_0$, and hence, for all λ with $\operatorname{Re}\lambda > \sigma_a$, by the uniqueness of analytic continuation.

It is known (cf. [**1**, Theorem 2.1]) that when $T(\cdot)$ is a semigroup of positive operators on an ordered Banach space, $R_s(\cdot)$ is analytic on $\{\lambda \in \mathbf{C};\ \operatorname{Re}\lambda > \sigma\}$, so that $w \leqslant \sigma_a = \sigma$. As will be seen in Proposition 7, this actually holds for all semigroups of class $(0, A)$.

By a semigroup of class $(0, A)$ we mean that $T(\cdot)$ is locally integrable and $\lambda R(\lambda)$ converges strongly to I as $\lambda \to \infty$. In this case, the operator A^0: $x \to \lim_{t\to 0} t^{-1}(T(t) - I)x$ is densely defined and closable (cf. [**2**, pp. 342–344]). The closure A of A^0 is called the infinitesimal generator of $T(\cdot)$. $T(\cdot)$ is of class (C_0) if $T(t)$ converges strongly to I as $t \to 0^+$. In this case, $T(\cdot)$ is also of class $(0, A)$ and A^0 is closed (see [**2**, p. 347]).

Let $S(t)$: $x \to \int_0^t T(s) x\, ds$ $(x \in X)$. The operators $C(t) := t^{-1} S(t)$, $t > 0$, are the Cesàro averages of $T(\cdot)$, and the operators $A(\lambda) := \lambda R(\lambda)$, $\lambda > \sigma_a$, are the Abel averages of $T(\cdot)$. Uniform ergodic theorems are concerned with the uniform operator convergence of $C(t)$ as $t \to \infty$ and of $A(\lambda)$ as $\lambda \to 0^+$. The result of Hille and Phillips [**2**, Theorem 18.8.4] deals with the uniform Abel-ergodicity of semigroups of class (A) (a class slightly larger than $(0, A)$) under the assumption "$w_0 \leqslant 0$". The theorem of Lin [**3**] treats uniform Cesàro-ergodicity and Abel-ergodicity for semigroups of class (C_0) under the assumption "$\lim_{t\to\infty} \|T(t)\|/t = 0$", which also implies $w_0 \leqslant 0$. However, as will be shown by an example in §3, it is possible for a semigroup to be uniformly ergodic while $-\infty = w = \sigma = \sigma_a < 0 < w_0$. Thus these well-known theorems do not apply universally.

The main purpose of this paper is to establish in §3 a uniform ergodic theorem (Theorem 4) for a general locally integrable semigroup, assuming the weaker condition "$\sigma_a \leqslant 0$". Under this assumption, the condition "$\lim_{t\to\infty} \|T(t)R(1)\|/t = 0$" (but not $\lim_{t\to\infty} \|T(t)\|/t = 0$) is necessary for uniform Cesàro-ergodicity of $T(\cdot)$. This is unlike the discrete case, where $\|T^n\|/n \to 0$ is necessary for the uniform ergodicity of $\{T^n\}$. When $T(\cdot)$ is a semigroup of class $(0, A)$, the uniform

Cesàro-ergodicity can be characterized precisely (Theorem 6). We shall begin with a uniform ergodic theorem (Theorem 1) for a general pseudo-resolvent. It will apply in §3 to provide conditions for uniform Abel-ergodicity of $T(\cdot)$.

2. Uniform ergodic theorems for pseudo-resolvents. In this section $R(\cdot)$ is a general pseudo-resolvent on an open subset Ω of the complex plane $\mathbf{C}$.

THEOREM 1. *Suppose that* $0 \in \overline{\Omega}$. *Then the following two statements are equivalent*:

(1) *There is a* $P \in B(X)$ *such that* $\|\lambda R(\lambda) - P\| \to 0$ *as* $\lambda \to 0$, $\lambda \in \Omega$.

(2) $\|\lambda^2 R(\lambda)\| \to 0$ *as* $\lambda \to 0$, *and the range* $\mathbf{R}(\lambda R(\lambda) - I)$ *of* $\lambda R(\lambda) - I$ *is closed for some* (*and hence all*) $\lambda \in \Omega$.

PROOF. Note first that the range $\mathbf{R}(\lambda R(\lambda) - I)$ and the null space $\mathbf{N}(\lambda R(\lambda) - I)$ of $\lambda R(\lambda) - I$ are independent of λ (see [**6**, p. 215]).

(1) $\Rightarrow$ (2). It is known from the mean ergodic theorem [**6**, p. 217] that P is the projection onto $\mathbf{N}(\lambda R(\lambda) - I)$ along $\overline{\mathbf{R}(\lambda R(\lambda) - I)}$. The fact that

$$\|\lambda R(\lambda) \,|\, \mathbf{N}(P)\| = \|(\lambda R(\lambda) - P) \,|\, \mathbf{N}(P)\| \leqslant \|\lambda R(\lambda) - P\| \to 0$$

as $\lambda \to 0$ implies that $(\lambda R(\lambda) - I) \,|\, \mathbf{N}(P)$ is invertible for small λ so that we have $\mathbf{R}(\lambda R(\lambda) - I) \supset \mathbf{N}(P) = \overline{\mathbf{R}(\lambda R(\lambda) - I)}$, i.e., $\mathbf{R}(\lambda R(\lambda) - I)$ is closed.

(2) $\Rightarrow$ (1). Fix a $\mu \neq 0$. Since $\mu R(\mu) - I$ has closed range, there exists a $M > 0$ such that each y in $\mathbf{R}(\mu R(\mu) - I)$ can be written as $y = (\mu R(\mu) - I)x$ for some x satisfying $\|x\| \leqslant M\|y\|$. Using the resolvent equation we have

$$\begin{aligned}\|\lambda R(\lambda) y\| &= \|\lambda R(\lambda)(\mu R(\mu) - I)x\| = \left\|(\mu - \lambda)^{-1}\left[\lambda^2 R(\lambda) - \lambda\mu R(\mu)\right]x\right\| \\ &\leqslant |\mu - \lambda|^{-1}\left[\|\lambda^2 R(\lambda)\| + |\lambda|\|\mu R(\mu)\|\right] M\|y\|,\end{aligned}$$

which implies that $\|\lambda R(\lambda) \,|\, \mathbf{R}(\lambda R(\lambda) - I)\| \to 0$ as $\lambda \to 0$. Hence for small λ the operator $K := (\lambda R(\lambda) - I) \,|\, \mathbf{R}(\lambda R(\lambda) - I)$ is invertible and so we have that $\mathbf{R}(\lambda R(\lambda) - I) = \mathbf{R}(K) = \mathbf{R}((\lambda R(\lambda) - I)^2)$. From this one easily deduces that $X = \mathbf{N}(\lambda R(\lambda) - I) + \mathbf{R}(\lambda R(\lambda) - I)$.

To show that the summation is direct, let y be in $\mathbf{N}(\lambda R(\lambda) - I) \cap \mathbf{R}(\lambda R(\lambda) - I)$. Then $y = \lambda R(\lambda) y$ for all $\lambda \in \Omega$ and, as shown above, $\lambda R(\lambda) y \to 0$ as $\lambda \to 0$. Hence $y = 0$ and so $X = \mathbf{N}(\lambda R(\lambda) - I) \oplus \mathbf{R}(\lambda R(\lambda) - I)$. Let P be the projection onto $\mathbf{N}(\lambda R(\lambda) - I)$ along $\mathbf{R}(\lambda R(\lambda) - I)$. Clearly we have $\|\lambda R(\lambda) - P\| = \|0 \oplus [\lambda R(\lambda) \,|\, \mathbf{R}(\lambda R(\lambda) - I)]\| \to 0$ as $\lambda \to 0$.

COROLLARY 2. *Suppose that the domain* Ω *of* $R(\cdot)$ *is unbounded. Then the following statements are equivalent*:

(1) *There is a* $Q \in B(X)$ *such that* $\|\lambda R(\lambda) - Q\| \to 0$ *as* $|\lambda| \to \infty$, $\lambda \in \Omega$.

(2) $\|R(\lambda)\| \to 0$ *as* $|\lambda| \to \infty$, *and* $\mathbf{R}(R(\lambda))$ *is closed for some* (*and hence all*) $\lambda \in \Omega$.

(3) $R(\lambda) = Q(\lambda I - A)^{-1}$ *for some* Q, $A \in B(X)$ *satisfying* $Q^2 = Q$, $AQ = QA = A$.

PROOF. Define $R_1(\lambda) := 1/\lambda - (1/\lambda^2)R(1/\lambda)$ for $\lambda \in \Omega_1 := \{\lambda \in C; 1/\lambda \in \Omega\}$. An easy computation shows that $R_1(\cdot)$ is a pseudo-resolvent on Ω_1, which has the limit point 0. Therefore, Theorem 1 applies to $R_1(\cdot)$, and the equivalence of (1) and

(2) follows by the facts that $\|\lambda^2 R_1(\lambda)\| \to 0$ as $\lambda \to 0$, $\lambda \in \Omega_1$, if and only if $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$, $\lambda \in \Omega$, and that $\|\lambda R_1(\lambda) - P\| \to 0$ as $\lambda \to 0$ if and only if $\|\lambda R(\lambda) - Q\| \to 0$ as $|\lambda| \to \infty$, where $Q = I - P$. "(3) $\Rightarrow$ (1)" is obvious, and "(1) $\Rightarrow$ (3)" is proved in Theorem 18.8.2 of [**2**].

REMARK. The operators P and Q turn out to be the linear projections with $\mathbf{R}(P) = \mathbf{N}(\lambda R(\lambda) - I)$, $\mathbf{N}(P) = \mathbf{R}(\lambda R(\lambda) - I)$, $\mathbf{R}(Q) = \mathbf{R}(R(\lambda))$ and $\mathbf{N}(Q) = \mathbf{N}(R(\lambda))$ for all $\lambda \in \Omega$.

3. Uniform ergodic theorems for locally integrable semigroups. All well-known uniform ergodic theorems for semigroups have been formulated for those of type $w_0 \leqslant 0$. We shall first give an example of a uniformly ergodic semigroup of class (C_0) which satisfies $-\infty = \sigma_a < 0 < w_0$, and then prove a uniform ergodic theorem for general locally integrable semigroups under the assumption $\sigma_a \leqslant 0$.

EXAMPLE. Let $1 \leqslant p < q < \infty$, and let X be the set of all Lebesgue measurable functions f on $(0, \infty)$ such that

$$\|f\| := \left(\int_0^\infty e^{ps^2} |f(s)|^p \, ds\right)^{1/p} + \left(\int_0^\infty |f(s)|^q \, ds\right)^{1/q} < \infty.$$

Then $(X, \|\cdot\|)$ is a Banach lattice which is reflexive whenever $p > 1$. For $\alpha \geqslant 0$ let $T_\alpha(\cdot)$ be the semigroup defined by $(T_\alpha(t)f)(s) := e^{\alpha t} f(t+s)$ $(f \in X, s, t \geqslant 0)$. Then $T_\alpha(t) = e^{\alpha(t)} T_0(t)$.

It was shown in [**1**] that $\|T_0(t)\| = 1$ for all $t \geqslant 0$ and for $T_0(\cdot)$ $w = \sigma_a = \sigma = -\infty$. Hence $T_\alpha(\cdot)$ has type $w_0 = \alpha$ and $w = \sigma_a = \sigma = -\infty$ for all $\alpha \geqslant 0$. Thus the infinitesimal generator $A_\alpha = d/ds + \alpha I$ of $T_\alpha(\cdot)$ has empty spectrum, and so

$$\|A(\lambda)\| = \left\|\lambda(\lambda I - A_\alpha)^{-1}\right\| \to 0 \cdot \left\|A_\alpha^{-1}\right\| = 0 \quad \text{as } \lambda \to 0.$$

Since $T_\alpha(t)$ are positive operators, we have for any nonnegative function f in X

$$C(t)f = t^{-1}\int_0^t T_\alpha(s) f \, ds \leqslant t^{-1}\int_0^t e^{1-s/t} T_\alpha(s) f \, ds$$
$$\leqslant t^{-1} e \int_0^\infty e^{-s/t} T_\alpha(s) f \, ds = e t^{-1} R(t^{-1}) f = e A(t^{-1}) f.$$

It follows that $\|C(t)\| \leqslant e\|A(t^{-1})\| \to 0$ as $t \to \infty$ (cf. [**4**, pp. 81 and 230]). Hence $T_\alpha(\cdot)$ is uniformly Abel and Cesàro ergodic to 0.

We shall need the following lemma in the proof of Theorem 4:

LEMMA 3. *Let $T(\cdot)$ be a locally integrable semigroup, and let $S(t)$ and $R_s(\lambda)$ be as defined in* §1. *The equality $(T(t) - I)R_s(\lambda) = S(t)(\lambda R_s(\lambda) - I)$ holds for all $t > 0$ and* $\operatorname{Re}\lambda > \sigma$.

PROOF. Integration by parts gives that

$$R_s(\lambda)x = \int_0^\infty e^{-\lambda u} T(u) x \, du = \lambda \int_0^\infty e^{-\lambda u} S(u) x \, du.$$

Then we use the identity $(T(t) - I)S(u) = S(t)(T(u) - I)$ (see [**5**]) to obtain that

$$(T(t) - I)R_s(\lambda)x = \lambda \int_0^\infty e^{-\lambda u}(T(t) - I)S(u) x \, du$$
$$= \lambda \int_0^\infty e^{-\lambda u} S(t)(T(u) - I) x \, du = S(t)(\lambda R_s(\lambda) - I)x.$$

Theorem 4. *Let $T(\cdot)$ be a locally integrable semigroup. Assume that $\sigma_a \leqslant 0$. Then the following statements are equivalent*:

(1) *$T(\cdot)$ is uniformly Cesàro-ergodic.*

(2) *$\|T(t)R(1)\|/t \to 0$ as $t \to \infty$, and $T(\cdot)$ is uniformly Abel-ergodic.*

(3) *$\|T(t)R(1)\|/t \to 0$ as $t \to \infty$, and $\mathbf{R}(R(1) - I)$ is closed.*

Proof. (1) $\Rightarrow$ (2). We have for each $x \in X$ and $a > 0$,

$$\begin{aligned}\|(A(\lambda) - P)x\| &= \left\|\lambda^2 \int_0^\infty e^{-\lambda t}(S(t) - tP)x\,dt\right\|\\ &\leqslant \left[\lambda^2 \int_0^a e^{-\lambda t}(\|S(t)\| + t\|P\|)\,dt + \lambda^2 \int_a^\infty e^{-\lambda t} t\|C(t) - P\|\,dt\right]\|x\|\\ &\leqslant \left[\left(\sup_{0 \leqslant t \leqslant a}\|S(t)\| + a\|P\|\right)\lambda^2 a + \sup_{t > a}\|C(t) - P\|\right]\|x\|.\end{aligned}$$

If $\|C(t) - P\| \to 0$ as $t \to \infty$, then it is easy to see from the above estimate that $\|A(\lambda) - P\| \to 0$ as $\lambda \to 0^+$. Then the fact that $\mathbf{N}(P) = \mathbf{R}(R(1) - I)$ (see the remark in §2) and Lemma 3 imply

$$\begin{aligned}\|(T(t) - I)R(1)\|/t &= \|C(t)(R(1) - I)\| = \|(C(t) - P)(R(1) - I)\|\\ &\leqslant \|C(t) - P\|\|R(1) - I\| \to 0 \qquad (t \to \infty).\end{aligned}$$

Hence the statement (2) holds when (1) holds.

"(2) $\Rightarrow$ (3)" is contained in Theorem 1.

(3) $\Rightarrow$ (1). First we prove that $\lim_{t \to \infty}\|T(t)R(1)\|/t = 0$ implies $\lim_{\lambda \to 0^+}\|\lambda^2 R(\lambda)\| = 0$. Given $\varepsilon > 0$, let $a > 0$ be such that $\|T(t)R(1)\| \leqslant \varepsilon t$ for all $t > a$. Using the resolvent equation we have for every $x \in X$

$$\begin{aligned}\|\lambda^2 R(\lambda)x\| &= \|\lambda^2[R(1) + (1 - \lambda)R(\lambda)R(1)]x\|\\ &\leqslant \lambda^2\|R(1)\|\|x\| + |1 - \lambda|\lambda^2 \int_0^\infty e^{-\lambda t}\|T(t)R(1)x\|\,dt\\ &\leqslant \lambda^2\|R(1)\|\|x\| + |1 - \lambda|\left[\lambda^2 \int_0^a \|T(t)R(1)x\|\,dt + \varepsilon\lambda^2 \int_a^\infty e^{-\lambda t} t\,dt\|x\|\right]\\ &\leqslant \left\{\lambda^2\|R(1)\| + |1 - \lambda|\left[\lambda^2\|W(a)\|\|R(1)\| + \varepsilon\right]\right\}\|x\|,\end{aligned}$$

where $W(a)$ denotes the operator from X to $L_1(X, [0, a])$ defined by $W(a)x = T(\cdot)x$, which is known to be bounded (cf. [**2**, p. 58]). It is easily seen from the above estimate that $\|\lambda^2 R(\lambda)\| \to 0$ as $\lambda \to 0^+$.

Now the statement (2) of Theorem 1 holds, and it was proved there that

$$X = \mathbf{N}(R(1) - I) \oplus \mathbf{R}(R(1) - I).$$

Let K be the restriction of $R(1) - I$ to $\mathbf{R}(R(1) - I)$. Then K is one-to-one, onto, and hence invertible. For $x \in \mathbf{R}(R(1) - I)$ let $y = K^{-1}x$. By Lemma 3 we have

$$\begin{aligned}\|C(t)x\| &= \|C(t)(R(1) - I)y\| = \|t^{-1}(T(t) - I)R(1)K^{-1}x\|\\ &\leqslant t^{-1}(\|T(t)R(1)\| + \|R(1)\|)\|K^{-1}\|\|x\|.\end{aligned}$$

This shows that $\|C(t)\,|\,\mathbf{N}(R(1)-I)\| \to 0$ as $t \to \infty$. Let P be the projection onto $\mathbf{R}(R(1)-I)$ along $\mathbf{R}(R(1)-I)$. In order to prove that $\|C(t)-P\|$ tends to 0 as $t \to \infty$, it remains to show that the restriction of $C(t)$ to $\mathbf{N}(R(1)-I)$ is an identity map for all $t > 0$. It suffices to show that $\mathbf{N}(R(1)-I) \subset \mathbf{N}(T(t)-I)$. Let $x \in \mathbf{N}(R(1)-I)$. Then $x \in \mathbf{N}(\lambda R(\lambda)-I)$ for all $\lambda > \sigma_a$ [6, p. 215], so that

$$\begin{aligned}(T(t)-I)x &= \lambda(T(t)-I)R(\lambda)x \\ &= \lambda S(t)(\lambda R(\lambda)-I)x = S(t)(\lambda^2 R(\lambda)-\lambda)x,\end{aligned}$$

which converges to 0 as $\lambda \to 0^+$. Hence x belongs to $\mathbf{N}(T(t)-I)$. The proof is now completed.

COROLLARY 5. *Let $T(\cdot)$ be a locally integrable semigroup satisfying $\sigma_a \leqslant 0$ and $\|T(t)R(1)\|/t \to 0$ $(t \to \infty)$. Then the following statements are equivalent*:

(1) *$T(\cdot)$ is uniformly Cesàro-ergodic.*

(2) *$T(\cdot)$ is uniformly Abel-ergodic.*

(3) *$\mathbf{R}(R(1)-I)$ is closed.*

REMARKS. (1) If $T(\cdot)$ is of class $(0, A)$ with generator A, one has that $\mathbf{R}(R(1)-I) = \mathbf{R}((I-A)^{-1}-I) = \mathbf{R}(A(I-A)^{-1}) = \mathbf{R}(A)$. Thus the theorem of Lin [3] is a specialization of Corollary 5.

(2) It follows from Theorem 4 that the semigroup $T_\alpha(\cdot)$ in the previous example satisfies $\|T_\alpha(t)R(1)\|/t \to 0$ $(t \to \infty)$, while $\|T_\alpha(t)\|/t = e^{\alpha t}t \to \infty$ in case $\alpha > 0$. Therefore, the hypothesis in Corollary 5 is in general strictly weaker than Lin's $(\|T(t)\|/t \to 0)$, and it cannot be further weakened.

The next theorem gives a precise characterization for the uniform Cesàro-ergodicity of $(0, A)$ semigroups.

THEOREM 6. *Let $T(\cdot)$ be a semigroup of class $(0, A)$. Then $T(\cdot)$ is uniformly Cesàro ergodic if and only if* (i) $\sigma \leqslant 0$, (ii) $\mathbf{R}(A)$ *is closed, and* (iii) $\|T(t)R(1)\|/t \to 0$ *as* $t \to \infty$.

This theorem is deduced from Theorem 4 and the following two propositions.

PROPOSITION 7. *If $T(\cdot)$ is a semigroup of class $(0, A)$, then $\sigma = \sigma_a$.*

PROOF. Since $\sigma \leqslant \sigma_a$, it suffices to show that if $\operatorname{Re}\lambda > \sigma$, then $\lambda I - A$ is invertible and $R_s(\lambda) = (\lambda I - A)^{-1}$.

We first prove that $(\lambda I - A^0)R_s(\lambda)x = x$ for $x \in D(A^0)$. Let

$$A_h = h^{-1}(T(h) - I).$$

Then we have

$$\begin{aligned}A_h R_s(\lambda)x &= A_h \lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\,ds \\ &= h^{-1}\lim_{t\to\infty}\int_0^t e^{-\lambda s}[T(s+h)x - T(s)x]\,ds \\ &= h^{-1}(e^{\lambda h}-1)R_s(\lambda)x - h^{-1}\int_0^h e^{\lambda(h-s)}T(s)x\,ds,\end{aligned}$$

which tends to $\lambda R_s(\lambda)x - x$ as $h \to 0$ if $x \in D(A^0)$, by the continuity of $T(t)x$ at $t = 0$. Hence $R_s(\lambda)x \in D(A^0)$ and $A^0 R_s(\lambda)x = \lambda R_s(\lambda)x - x$.

If $y \in X$, we can obtain a sequence $y_n \in D(A^0)$ such that $y_n \to y$. Then we have that $R_s(\lambda)y_n \to R_s(\lambda)y$ and $A^0 R_s(\lambda)y_n - y_n = \lambda R_s(\lambda)y_n - y_n \to \lambda R_s(\lambda)y - y$. Since A is the closure of A^0, we have proved that $R_s(\lambda)X \subset D(A)$ and $(\lambda I - A)R_s(\lambda) = I$.

It remains to show that $AR_s(\lambda)x = R_s(\lambda)Ax$ for $x \in D(A)$. Given $x \in D(A)$ there exists a sequence $x_n \in D(A^0)$ such that $x_n \to x$ and $A^0 x_n \to Ax$. Since $A_h R_s(\lambda)x_n = R_s(\lambda)A_h x_n$, by letting $h \to 0$ we obtain that $A^0 R_s(\lambda)x_n = R_s(\lambda)A^0 x_n$. Therefore, $A^0 R_s(\lambda)x_n$ tends to $R_s(\lambda)Ax$ as $n \to \infty$. This and the fact that $R_s(\lambda)x_n \to R_s(\lambda)x$ show that $R_s(\lambda)x \in D(A)$ and $AR_s(\lambda)x = R_s(\lambda)Ax$.

PROPOSITION 8. *If a locally integrable semigroup $T(\cdot)$ is strongly Cesàro-ergodic, then $\sigma \leqslant 0$, i.e. the Laplace transform $R_s(\lambda)$ exists for all λ with* $\operatorname{Re}\lambda > 0$.

PROOF. The uniform boundedness principle implies that $\|C(t)\| \leqslant M$ for all $t > 1$. Let $\operatorname{Re}\lambda > 0$. We have for all $v > u > 0$ and $x \in X$

$$\left\|\int_u^v e^{-\lambda t}T(t)x\,dt\right\| = \left\|e^{-\lambda t}S(t)x\,|_u^v + \lambda\int_u^v e^{-\lambda t}S(t)x\,dt\right\|$$
$$\leqslant \left\{|e^{-\lambda v}|v + |e^{-\lambda u}|u + |\lambda|\int_u^v e^{-t\operatorname{Re}\lambda}t\,dt\right\}M\|x\|,$$

which shows that $\|\int_u^v e^{-\lambda t}T(t)\,dt\| \to 0$ as $u \to \infty$. Hence

$$R_s(\lambda) = \text{uo-}\lim_{t\to\infty}\int_0^u e^{-\lambda t}T(t)\,dt$$

exists.

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