UNIFORM ERGODIC THEOREMS
FOR LOCALLY INTEGRABLE SEMIGROUPS
AND PSEUDO-RESOLVENTS

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Abstract. We study uniform ergodicity (at \( \infty \)) of a locally integrable operator semigroup \( T(\cdot) \) of type \( w_0 \) under a suitable condition which is weaker than the usual one "\( w_0 \leq 0 \)". We also give a precise characterization of the uniform Cesàro-ergodicity for semigroups of class \((0, A)\). To prove the part of Abel-ergodicity we first prove a general uniform ergodic theorem for pseudo-resolvents.

1. Introduction. Let \( B(X) \) be the Banach algebra of all bounded linear operators on a Banach space \( X \), and let \( \{T(t); t > 0\} \) be a family in \( B(X) \) with the properties:

(i) \( T(s + t) = T(s)T(t) \) for all \( s, t > 0 \); (ii) for each \( x \in X \) the function \( T(\cdot)x \) is Bochner integrable with respect to the Lebesgue measure over every finite subinterval of \((0, \infty)\). Such a family \( T(\cdot) \) is called a locally integrable semigroup. It is known to be strongly continuous on \((0, \infty)\).

The type of \( T(\cdot) \) is the number \( w_0 := \inf_{t>0} t^{-1} \log \|T(t)\| < \infty \). For every \( \lambda \) with \( \text{Re} \lambda > w_0 \) and \( x \in X \) the Bochner integral

\[ R(\lambda)x := \int_0^\infty e^{-\lambda t}T(t)x \, dt \]

exists and defines a bounded linear operator \( R(\lambda) \) on \( X \). The function \( R(\lambda) \), \( \text{Re} \lambda > w_0 \) (called the Laplace transform of \( T(\cdot) \)), satisfies the first resolvent equation

\[ R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu), \]

that is, \( R(\cdot) \) is a pseudo-resolvent on \( \{ \lambda \in \mathbb{C}; \text{Re} \lambda > w_0 \} \) (cf. [2, p. 510]). \( R(\cdot) \) has a unique maximal extension satisfying the above equation, and the domain of definition \( \Omega \) of this maximally extended pseudo-resolvent, which we shall still denote by \( R(\cdot) \), is an open subset of the complex plane \( \mathbb{C} \), on which \( R(\cdot) \) is analytic and cannot be continued analytically beyond (cf. [2, pp. 188–189]). We denote \( w := \inf\{ u \in (-\infty, \infty); \lambda \in \Omega \text{ for all } \lambda \text{ with } \text{Re} \lambda > u \} \). Then \( w \leq w_0 \).

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One can also define the Laplace transform $R_s(\lambda)$ of $T(\lambda)$ in the following weaker sense:

$$R_s(\lambda) := \text{so- lim}_{t \to \infty} \int_0^t e^{-\lambda s} T(s) \, ds,$$

where the integral represents the operator which maps each $x \in X$ to the corresponding Bochner integral $\int_0^\infty e^{-\lambda s} T(s)x \, ds$. It can be proved (cf. [1, p. 248]) that if for a number $\lambda_0$ $R_s(\lambda_0)$ exists, then $R_s(\lambda)$ exists for all $\lambda$ with $\text{Re}\lambda > \text{Re}\lambda_0$. Let us denote

$$\sigma := \inf\{ u \in (-\infty, \infty); R_s(\lambda) \text{ exists for all } \lambda \text{ with } \text{Re}\lambda > u\},$$

$$\sigma_a := \inf\{ u \in (-\infty, \infty); R_s(u) \text{ is analytic for all } \lambda \text{ with } \text{Re}\lambda > u\}.$$

It is clear that $\sigma \leq \sigma_a$, $\omega \leq \sigma_a \leq \omega_0$, and that $R_s(\lambda) = R(\lambda)$ for all $\lambda$ with $\text{Re}\lambda > \omega_0$, and hence, for all $\lambda$ with $\text{Re}\lambda > \sigma_a$, by the uniqueness of analytic continuation.

It is known (cf. [1, Theorem 2.1]) that when $T(\lambda)$ is a semigroup of positive operators on an ordered Banach space, $R_s(\lambda)$ is analytic on $\{ \lambda \in \mathbb{C}; \text{Re}\lambda > \sigma\}$, so that $\omega \leq \sigma_a = \sigma$. As will be seen in Proposition 7, this actually holds for all semigroups of class $(0, A)$.

By a semigroup of class $(0, A)$ we mean that $T(\lambda)$ is locally integrable and $\lambda R(\lambda)$ converges strongly to $I$ as $\lambda \to \infty$. In this case, the operator $A^0: x \to \lim_{t \to 0} t^{-1}(T(t) - I)x$ is densely defined and closable (cf. [2, pp. 342–344]). The closure $A$ of $A^0$ is called the infinitesimal generator of $T(\cdot)$. $T(\cdot)$ is of class $(C_0)$ if $T(t)$ converges strongly to $I$ as $t \to 0^+$. In this case, $T(\cdot)$ is also of class $(0, A)$ and $A^0$ is closed (see [2, p. 347]).

Let $S(t): x \to \int_0^t T(s)x \, ds$ ($x \in X$). The operators $C(t) := t^{-1}S(t)$, $t > 0$, are the Cesàro averages of $T(\cdot)$, and the operators $A(x) := \lambda R(\lambda)$, $\lambda > \sigma_a$, are the Abel averages of $T(\cdot)$. Uniform ergodic theorems are concerned with the uniform operator convergence of $C(t)$ as $t \to \infty$ and of $A(\lambda)$ as $\lambda \to 0^+$. The result of Hille and Phillips [2, Theorem 18.8.4] deals with the uniform Abel-ergodicity of semigroups of class $(A)$ (a class slightly larger than $(0, A)$) under the assumption “$\omega_0 \leq 0$”. The theorem of Lin [3] treats uniform Cesàro-ergodicity and Abel-ergodicity for semigroups of class $(C_0)$ under the assumption “$\lim_{t \to \infty} \|T(t)\|/t = 0$”, which also implies $\omega_0 \leq 0$. However, as will be shown by an example in §3, it is possible for a semigroup to be uniformly ergodic while $-\infty = w = \sigma = \sigma_a < 0 < \omega_0$. Thus these well-known theorems do not apply universally.

The main purpose of this paper is to establish in §3 a uniform ergodic theorem (Theorem 4) for a general locally integrable semigroup, assuming the weaker condition “$\sigma_a \leq 0$”. Under this assumption, the condition “$\lim_{t \to \infty} \|T(t)R(1)\|/t = 0$” (but not $\lim_{t \to \infty} \|T(t)\|/t = 0$) is necessary for uniform Cesàro-ergodicity of $T(\cdot)$. This is unlike the discrete case, where $\|T^n\|/n \to 0$ is necessary for the uniform ergodicity of $\{T^n\}$. When $T(\cdot)$ is a semigroup of class $(0, A)$, the uniform
Cesàro-ergodicity can be characterized precisely (Theorem 6). We shall begin with a uniform ergodic theorem (Theorem 1) for a general pseudo-resolvent. It will apply in §3 to provide conditions for uniform Abel-ergodicity of $T(\cdot)$.

2. Uniform ergodic theorems for pseudo-resolvents. In this section $R(\cdot)$ is a general pseudo-resolvent on an open subset $\Omega$ of the complex plane $C$.

**Theorem 1.** Suppose that $0 \in \overline{\Omega}$. Then the following two statements are equivalent:

1. There is a $P \in B(X)$ such that $\|\lambda R(\lambda) - P\| \to 0$ as $\lambda \to 0$, $\lambda \in \Omega$.
2. $\|\lambda^2 R(\lambda)\| \to 0$ as $\lambda \to 0$, and the range $R(\lambda R(\lambda) - I)$ of $\lambda R(\lambda) - I$ is closed for some (and hence all) $\lambda \in \Omega$.

**Proof.** Note first that the range $R(\lambda R(\lambda) - I)$ and the null space $N(\lambda R(\lambda) - I)$ of $\lambda R(\lambda) - I$ are independent of $\lambda$ (see [6, p. 215]).

(1) $\Rightarrow$ (2). It is known from the mean ergodic theorem [6, p. 217] that $P$ is the projection onto $N(\lambda R(\lambda) - I)$ along $R(\lambda R(\lambda) - I)$. The fact that $\|\lambda R(\lambda)\|N(P)\| = \|R(\lambda R(\lambda) - I)\|N(P)\| \leq \|\lambda R(\lambda) - P\| \to 0$ as $\lambda \to 0$ implies that $(\lambda R(\lambda) - I)|N(P)$ is invertible for small $\lambda$ so that we have $R(\lambda R(\lambda) - I) \supset N(P) = R(\lambda R(\lambda) - I)$, i.e., $R(\lambda R(\lambda) - I)$ is closed.

(2) $\Rightarrow$ (1). Fix a $\mu \neq 0$. Since $\mu R(\mu) - I$ has closed range, there exists a $M > 0$ such that each $y$ in $R(\mu R(\mu) - I)$ can be written as $y = (\mu R(\mu) - I)x$ for some $x$ satisfying $\|x\| \leq M\|y\|$. Using the resolvent equation we have

$$\|\lambda R(\lambda) y\| = \|\lambda R(\lambda)(\mu R(\mu) - I)x\| = \|(\mu - \lambda)^{-1}[\lambda^2 R(\lambda) - \lambda \mu R(\mu)]x\|$$

$$\leq |\mu - \lambda|^{-1}[\|\lambda^2 R(\lambda)\| + |\lambda|\|\mu R(\mu)\|]M\|y\|,$$

which implies that $\|\lambda R(\lambda)R(\lambda R(\lambda) - I)\| \to 0$ as $\lambda \to 0$. Hence for small $\lambda$ the operator $K := (\lambda R(\lambda) - I)R(\lambda R(\lambda) - I)$ is invertible and so we have that $R(\lambda R(\lambda) - I) = R(K) = R(\lambda R(\lambda) - I^2)$. From this one easily deduces that $X = N(\lambda R(\lambda) - I) + R(\lambda R(\lambda) - I)$.

To show that the summation is direct, let $y$ be in $N(\lambda R(\lambda) - I) \cap R(\lambda R(\lambda) - I)$. Then $y = \lambda R(\lambda)y$ for all $\lambda \in \Omega$ and, as shown above, $\lambda R(\lambda)y \to 0$ as $\lambda \to 0$. Hence $y = 0$ and so $X = N(\lambda R(\lambda) - I) \oplus R(\lambda R(\lambda) - I)$. Let $P$ be the projection onto $N(\lambda R(\lambda) - I)$ along $R(\lambda R(\lambda) - I)$. Clearly we have $\|\lambda R(\lambda) - P\| = \|0 \oplus [\lambda R(\lambda)R(\lambda R(\lambda) - I)]\| \to 0$ as $\lambda \to 0$.

**Corollary 2.** Suppose that the domain $\Omega$ of $R(\cdot)$ is unbounded. Then the following statements are equivalent:

1. There is a $Q \in B(X)$ such that $\|\lambda R(\lambda) - Q\| \to 0$ as $|\lambda| \to \infty$, $\lambda \in \Omega$.
2. $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$, and $R(R(\lambda))$ is closed for some (and hence all) $\lambda \in \Omega$.
3. $R(\lambda) = Q(\lambda I - A)^{-1}$ for some $Q, A \in B(X)$ satisfying $Q^2 = Q, QA = AQ = A$.

**Proof.** Define $R_1(\lambda) := 1/\lambda - (1/\lambda^2)R(1/\lambda)$ for $\lambda \in \Omega_1 := \{\lambda \in C; 1/\lambda \in \Omega\}$. An easy computation shows that $R_1(\cdot)$ is a pseudo-resolvent on $\Omega_1$, which has the limit point 0. Therefore, Theorem 1 applies to $R_1(\cdot)$, and the equivalence of (1) and
(2) follows by the facts that $||\lambda^2 R_1(\lambda)|| \to 0$ as $\lambda \to 0$, $\lambda \in \Omega$, if and only if $||R(\lambda)|| \to 0$ as $|\lambda| \to \infty$, $\lambda \in \Omega$, and that $||\lambda R(\lambda) - P|| \to 0$ as $\lambda \to 0$ if and only if $||\lambda R(\lambda) - Q|| \to 0$ as $|\lambda| \to \infty$, where $Q = I - P$. "(3) $\Rightarrow$ (1)" is obvious, and "(1) $\Rightarrow$ (3)" is proved in Theorem 18.8.2 of [2].

Remark. The operators $P$ and $Q$ turn out to be the linear projections with $R(P) = N(\lambda R(\lambda) - I)$, $N(P) = R(\lambda R(\lambda) - I)$, $R(Q) = R(R(\lambda))$ and $N(Q) = N(R(\lambda))$ for all $\lambda \in \Omega$.

3. Uniform ergodic theorems for locally integrable semigroups. All well-known uniform ergodic theorems for semigroups have been formulated for those of type $w_0 \leq 0$. We shall first give an example of a uniformly ergodic semigroup of class $(C_0)$ which satisfies $-\infty = \sigma_a < 0 < w_0$, and then prove a uniform ergodic theorem for general locally integrable semigroups under the assumption $\sigma_a \leq 0$.

Example. Let $1 < p < q < \infty$, and let $X$ be the set of all Lebesgue measurable functions $f$ on $(0, \infty)$ such that

$$
11f1 := (\int_0^\infty |f(s)|^p ds)^{1/p} + (\int_0^\infty |f(s)|^q ds)^{1/q} < \infty.
$$

Then $(X, || \cdot ||)$ is a Banach lattice which is reflexive whenever $p > 1$. For $\alpha \geq 0$ let $T_\alpha(\cdot)$ be the semigroup defined by $(T_\alpha(t)f)(s) := e^{\alpha t}f(t + s)$ ($f \in X$, $s, t \geq 0$). Then $T_\alpha(t) = e^{\alpha t}T_0(t)$.

It was shown in [1] that $||T_0(t)|| = 1$ for all $t \geq 0$ and for $T_0(\cdot) w = \sigma_a = \sigma = -\infty$. Hence $T_\alpha(\cdot)$ has type $w_0 = \alpha$ and $w = \sigma_a = \sigma = -\infty$ for all $\alpha \geq 0$. Thus the infinitesimal generator $A_\alpha = d/ds + \alpha I$ of $T_\alpha(\cdot)$ has empty spectrum, and so

$$
||A(\lambda)|| = ||\lambda(\lambda I - A_\alpha)^{-1}|| \to 0 \cdot ||A_\alpha^{-1}|| = 0 \quad \text{as } \lambda \to 0.
$$

Since $T_\alpha(t)$ are positive operators, we have for any nonnegative function $f$ in $X$

$$
C(t)f = t^{-1}\int_0^t T_\alpha(s)f ds \leq t^{-1}\int_0^t e^{-s/t}T_\alpha(s)f ds
$$

$$
\leq t^{-1}e^{\int_0^\infty e^{-s/t}T_\alpha(s)f ds} = e\int_0^\infty e^{-s/t}T_\alpha(s)f ds = e\int_0^\infty (t^{-1})f = eA(t^{-1})f.
$$

It follows that $||C(t)|| \leq e||A(t^{-1})|| \to 0$ as $t \to \infty$ (cf. [4, pp. 81 and 230]). Hence $T_\alpha(\cdot)$ is uniformly Abel and Cesàro ergodic to 0.

We shall need the following lemma in the proof of Theorem 4:

Lemma 3. Let $T(\cdot)$ be a locally integrable semigroup, and let $S(t)$ and $R_\gamma(\lambda)$ be as defined in §1. The equality $(T(t) - I)R_\gamma(\lambda) = S(t)(\lambda R_\gamma(\lambda) - I)$ holds for all $t > 0$ and $\Re \lambda > \sigma$.

Proof. Integration by parts gives that

$$
R_\gamma(\lambda)x = \int_0^\infty e^{-\lambda u}T(u)x du = \lambda\int_0^\infty e^{-\lambda u}S(u)x du.
$$

Then we use the identity $(T(t) - I)S(u) = S(t)(T(u) - I)$ (see [5]) to obtain that

$$(T(t) - I)R_\gamma(\lambda)x = \lambda\int_0^\infty e^{-\lambda u}(T(t) - I)S(u)x du
$$

$$
= \lambda\int_0^\infty e^{-\lambda u}S(t)(T(u) - I)x du = S(t)(\lambda R_\gamma(\lambda) - I)x.
$$
Theorem 4. Let $T(\cdot)$ be a locally integrable semigroup. Assume that $\sigma_a \leq 0$. Then the following statements are equivalent:

1. $T(\cdot)$ is uniformly Cesàro-ergodic.
2. $\|T(t)R(1)\|/t \to 0$ as $t \to \infty$, and $T(\cdot)$ is uniformly Abel-ergodic.
3. $\|T(t)R(1)\|/t \to 0$ as $t \to \infty$, and $\mathbb{R}(R(1) - I)$ is closed.

Proof. $(1) \Rightarrow (2)$. We have for each $x \in X$ and $a > 0$,

$$
\|A(\lambda) - P\|x = \left\| \lambda^2 \int_0^\infty e^{-\lambda t}(S(t) - tP)x \, dt \right\|
\leq \left[ \lambda^2 \int_0^a e^{-\lambda t}(\|S(t)\| + t\|P\|) \, dt + \lambda^2 \int_a^\infty e^{-\lambda t}\|C(t) - P\| \, dt \right]\|x\|
\leq \left( \sup_{0 < t < a} \|S(t)\| + a\|P\| \right)\lambda^2 a + \sup_{t > a} \|C(t) - P\|\|x\|.
$$

If $\|C(t) - P\| \to 0$ as $t \to \infty$, then it is easy to see from the above estimate that $\|A(\lambda) - P\| \to 0$ as $\lambda \to 0^+$. Then the fact that $\mathbb{N}(P) = \mathbb{R}(R(1) - I)$ (see the remark in §2) and Lemma 3 imply

$$
\|(T(t) - I)R(1)\|/t = \|C(t)(R(1) - I)\| = \|C(t) - P\|(R(1) - I)\| \to 0 \quad (t \to \infty).
$$

Hence the statement (2) holds when (1) holds.

"(2) ⇒ (3)" is contained in Theorem 1.

(3) ⇒ (1). First we prove that $\lim_{t \to \infty} \|T(t)R(1)\|/t = 0$ implies $\lim_{\lambda \to 0^+} \|\lambda^2 R(\lambda)\| = 0$. Given $\varepsilon > 0$, let $a > 0$ be such that $\|T(t)R(1)\| \leq \varepsilon t$ for all $t > a$. Using the resolvent equation we have for every $x \in X$

$$
\|\lambda^2 R(\lambda)\| = \|\lambda^2 [R(1) + (1 - \lambda)R(\lambda)R(1)]\|x\|
\leq \lambda^2 \|R(1)\||x|| + |1 - \lambda| \lambda^2 \int_0^\infty e^{-\lambda t}\|T(t)R(1)x\| \, dt
\leq \lambda^2 \|R(1)\||x|| + |1 - \lambda| \left[ \lambda^2 \int_0^a \|T(t)R(1)x\| \, dt + \varepsilon \lambda^2 \int_a^\infty e^{-\lambda t} \, dt\|x\| \right]
\leq \left\{ \lambda^2 \|R(1)\| + |1 - \lambda| \left[ \lambda^2 \|W(a)\||R(1)\| + \varepsilon \right] \right\} \|x\|,
$$

where $W(a)$ denotes the operator from $X$ to $L_1(X,[0,a])$ defined by $W(a)x = T(\cdot)x$, which is known to be bounded (cf. [2, p. 58]). It is easily seen from the above estimate that $\|\lambda^2 R(\lambda)\| \to 0$ as $\lambda \to 0^+$.

Now the statement (2) of Theorem 1 holds, and it was proved there that $X = \mathbb{N}(R(1) - I) \oplus \mathbb{R}(R(1) - I)$.

Let $K$ be the restriction of $R(1) - I$ to $\mathbb{R}(R(1) - I)$. Then $K$ is one-to-one, onto, and hence invertible. For $x \in \mathbb{R}(R(1) - I)$ let $y = K^{-1}x$. By Lemma 3 we have

$$
\|C(t)x\| = \|C(t)(R(1) - I)y\| = \|t^{-1}(T(t) - I)R(1)K^{-1}x\|
\leq t^{-1}(\|T(t)R(1)\| + \|R(1)\|)\|K^{-1}\|\|x\|.
$$
This shows that \(|C(t)|N(R(1) - I)| \to 0 as t \to \infty. Let P be the projection onto 
\(R(R(1) - I)\) along \(R(R(1) - I)\). In order to prove that \(|C(t)|P| \to 0 as t \to \infty\), it remains to show that the restriction of \(C(t)\) to \(N(R(1) - I)\) is an identity map for all \(t > 0\). It suffices to show that \(N(R(1) - I) \subseteq N(T(t) - I)\). Let \(x \in N(R(1) - I)\). Then \(x \in N(\lambda R(\lambda) - I)\) for all \(\lambda > \sigma\) \([6, p. 215]\), so that 
\[
(T(t) - I)x = \lambda(T(t) - I)R(\lambda)x \\
= \lambda S(t)(\lambda R(\lambda) - I)x = S(t)(\lambda^2 R(\lambda) - \lambda)x,
\]
which converges to 0 as \(\lambda \to 0^+\). Hence \(x\) belongs to \(N(T(t) - I)\). The proof is now completed.

**Corollary 5.** Let \(T(\cdot)\) be a locally integrable semigroup satisfying \(\sigma \leq 0\) and 
\(|T(t)R(1)|/t \to 0 (t \to \infty). Then the following statements are equivalent:

1. \(T(\cdot)\) is uniformly Cesàro-ergodic.
2. \(T(\cdot)\) is uniformly Abel-ergodic.
3. \(R(R(1) - I)\) is closed.

**Remarks.** (1) If \(T(\cdot)\) is of class \((0, A)\) with generator \(A\), one has that \(R(R(1) - I) = R((I - A)^{-1} - I) = R(A(I - A)^{-1}) = R(A)\). Thus the theorem of Lin \([3]\) is a specialization of Corollary 5.

(2) It follows from Theorem 4 that the semigroup \(T_a(\cdot)\) in the previous example satisfies 
\(|T_a(t)R(1)|/t \to 0 (t \to \infty)\), while \(|T_a(t)|/t = e^{\alpha_1 t} \to \infty\) in case \(\alpha > 0\). Therefore, the hypothesis in Corollary 5 is in general strictly weaker than Lin's 
\(||T(t)||/t \to 0)\), and it cannot be further weakened.

The next theorem gives a precise characterization for the uniform Cesàro-ergodicity of \((0, A)\) semigroups.

**Theorem 6.** Let \(T(\cdot)\) be a semigroup of class \((0, A)\). Then \(T(\cdot)\) is uniformly Cesàro 
ergodic if and only if (i) \(\sigma \leq 0\), (ii) \(R(A)\) is closed, and (iii) \(|T(t)R(1)|/t \to 0 as t \to \infty.\)

This theorem is deduced from Theorem 4 and the following two propositions.

**Proposition 7.** If \(T(\cdot)\) is a semigroup of class \((0, A)\), then \(\sigma = \sigma_a\).

**Proof.** Since \(\sigma \leq \sigma_a\), it suffices to show that if \(\text{Re} \lambda > \sigma\), then \(\lambda I - A\) is 
invertible and \(R_\delta(\lambda) = (\lambda I - A)^{-1}\).

We first prove that \((\lambda I - A^0)R_\delta(\lambda)x = x\) for \(x \in D(A^0)\). Let 
\[A_h = h^{-1}(T(h) - I)\]
Then we have 
\[
A_h R_\delta(\lambda)x = A_h \lim_{t \to \infty} \int_0^t e^{-\lambda s}T(s)x \, ds \\
= h^{-1} \lim_{t \to \infty} \int_0^t e^{-\lambda s} [T(s + h)x - T(s)x] \, ds \\
= h^{-1} (e^{\lambda h} - 1) R_\delta(\lambda)x - h^{-1} \int_0^h e^{\lambda (h-s)}T(s)x \, ds,
\]
which tends to $\lambda R_s(\lambda)x - x$ as $h \to 0$ if $x \in D(A^0)$, by the continuity of $T(t)x$ at $t = 0$. Hence $R_s(\lambda)x \in D(A^0)$ and $A^0R_s(\lambda)x = \lambda R_s(\lambda)x - x$.

If $y \in X$, we can obtain a sequence $y_n \in D(A^0)$ such that $y_n \to y$. Then we have that $R_s(\lambda)y_n \to R_s(\lambda)y$ and $A^0R_s(\lambda)y_n - y_n = \lambda R_s(\lambda)y_n - y_n \to \lambda R_s(\lambda)y - y$. Since $A$ is the closure of $A^0$, we have proved that $R_s(\lambda)X \subset D(A)$ and $(\lambda I - A)R_s(\lambda) = I$.

It remains to show that $AR_s(\lambda)x = R_s(\lambda)Ax$ for $x \in D(A)$. Given $x \in D(A)$ there exists a sequence $x_n \in D(A^0)$ such that $x_n \to x$ and $A^0x_n \to Ax$. Since $A^0R_s(\lambda)x_n = A^0R_s(\lambda)A_n x_n$, by letting $h \to 0$ we obtain that $A^0R_s(\lambda)x_n = A^0R_s(\lambda)A^0x_n$. Therefore, $A^0R_s(\lambda)x_n$ tends to $R_s(\lambda)Ax$ as $n \to \infty$. This and the fact that $R_s(\lambda)x_n \to R_s(\lambda)x$ show that $R_s(\lambda)x \in D(A)$ and $AR_s(\lambda)x = R_s(\lambda)Ax$.

**Proposition 8.** If a locally integrable semigroup $T(\cdot)$ is strongly Cesàro-ergodic, then $\sigma \leq 0$, i.e. the Laplace transform $R_s(\lambda)$ exists for all $\lambda$ with $\text{Re}\lambda > 0$.

**Proof.** The uniform boundedness principle implies that $||C(t)|| \leq M$ for all $t > 1$. Let $\text{Re}\lambda > 0$. We have for all $v > u > 0$ and $x \in X$

$$\left\| \int_u^v e^{\lambda t}T(t)x \, dt \right\| = \left\| e^{\lambda S(t)}x \int_u^v e^{-\lambda S(t)}x \, dt \right\| \leq \left\{ |e^{-\lambda u}| + |e^{-\lambda v}| + |x| \left( \int_u^v e^{-t\text{Re}\lambda} \, dt \right) M \right\} \|x\|,$$

which shows that $\|\int_u^v e^{-\lambda t}T(t) \, dt\| \to 0$ as $u \to \infty$. Hence

$$R_s(\lambda) = \text{uo-} \lim_{t \to \infty} \int_0^t e^{-\lambda t}T(t) \, dt$$

exists.

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**References**


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