UNIFORM ERGODIC THEOREMS 
FOR LOCALLY INTEGRABLE SEMIGROUPS 
AND PSEUDO-RESOLVENTS 

SEN - YEN SHAW 

Abstract. We study uniform ergodicity (at $\infty$) of a locally integrable operator 
semigroup $T(\cdot)$ of type $w_0$ under a suitable condition which is weaker than the usual 
one $"w_0 \leq 0"$. We also give a precise characterization of the uniform Cesàro-ergodic-
ity for semigroups of class $(0, A)$. To prove the part of Abel-ergodicity we first prove 
a general uniform ergodic theorem for pseudo-resolvents. 

1. Introduction. Let $B(X)$ be the Banach algebra of all bounded linear operators 
on a Banach space $X$, and let $\{T(t); t > 0\}$ be a family in $B(X)$ with the properties: 
(i) $T(s + t) = T(s)T(t)$ for all $s, t > 0$; (ii) for each $x \in X$ the function $T(\cdot)x$ is 
Bochner integrable with respect to the Lebesgue measure over every finite subinterval of $(0, \infty)$. Such a family $T(\cdot)$ is called a locally integrable semigroup. It is known 
to be strongly continuous on $(0, \infty)$. 

The type of $T(\cdot)$ is the number $w_0 := \inf_{t > 0} t^{-1}\log\|T(t)\| < \infty$. For every $\lambda$ with 
$\Re \lambda > w_0$ and $x \in X$ the Bochner integral 

$$R(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)x \, dt$$ 

exists and defines a bounded linear operator $R(\lambda)$ on $X$. The function $R(\lambda)$, 
$\Re \lambda > w_0$ (called the Laplace transform of $T(\cdot)$), satisfies the first resolvent 
equation 

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu),$$ 

that is, $R(\cdot)$ is a pseudo-resolvent on $\{\lambda \in \mathbb{C}; \Re \lambda > w_0\}$ (cf. [2, p. 510]). $R(\cdot)$ has 
a unique maximal extension satisfying the above equation, and the domain of 
definition $\Omega$ of this maximally extended pseudo-resolvent, which we shall still denote 
by $R(\cdot)$, is an open subset of the complex plane $\mathbb{C}$, on which $R(\cdot)$ is analytic and 
cannot be continued analytically beyond (cf. [2, pp. 188–189]). We denote 
$$w := \inf\{u \in (-\infty, \infty); \lambda \in \Omega \text{ for all } \lambda \text{ with } \Re \lambda > u\}.$$ Then $w \leq w_0$. 

Received by the editors August 7, 1984 and, in revised form, August 29, 1985. 
1980 Mathematics Subject Classification (1985 Revision). Primary 47A35, 47D05. 
Key words and phrases. Cesàro-ergodicity, Abel-ergodicity, locally integrable semigroup, Laplace 
transform, pseudo-resolvent.
One can also define the Laplace transform $R_s(\cdot)$ of $T(\cdot)$ in the following weaker sense:

$$R_s(\lambda) := \lim_{t \to \infty} \int_0^t e^{-\lambda s} T(s) \, ds,$$

where the integral represents the operator which maps each $x \in X$ to the corresponding Bochner integral $\int_0^t e^{-\lambda s} T(s) x \, ds$. It can be proved (cf. [1, p. 248]) that if for a number $\lambda_0$ $R_s(\lambda_0)$ exists, then $R_s(\lambda)$ exists for all $\lambda$ with $\text{Re} \lambda > \text{Re} \lambda_0$. Let us denote

$$\sigma := \inf \{ u \in (-\infty, \infty); R_s(\lambda) \text{ exists for all } \lambda \text{ with } \text{Re} \lambda > u \}$$

$$= \inf \{ u \in (-\infty, \infty); R_s(u) \text{ exists} \},$$

$$\sigma_a := \inf \{ u \in (-\infty, \infty); R_s(u) \text{ is analytic for all } \lambda \text{ with } \text{Re} \lambda > u \}.$$

It is clear that $\sigma \leq \sigma_a$, $\omega \leq \sigma_a \leq \omega_0$, and that $R_s(\lambda) = R(\lambda)$ for all $\lambda$ with $\text{Re} \lambda > \omega_0$, and hence, for all $\lambda$ with $\text{Re} \lambda > \sigma_a$, by the uniqueness of analytic continuation.

It is known (cf. [1, Theorem 2.1]) that when $T(\cdot)$ is a semigroup of positive operators on an ordered Banach space, $R_s(\cdot)$ is analytic on $\{ \lambda \in \mathbb{C}; \text{Re} \lambda > \sigma \}$, so that $\omega \leq \sigma_a = \sigma$. As will be seen in Proposition 7, this actually holds for all semigroups of class $(0, A)$.

By a semigroup of class $(0, A)$ we mean that $T(\cdot)$ is locally integrable and $\lambda R(\lambda)$ converges strongly to $I$ as $\lambda \to \infty$. In this case, the operator $A^0: x \to \lim_{r \to 0} r^{-1}(T(t) - I)x$ is densely defined and closable (cf. [2, pp. 342–344]). The closure $A$ of $A^0$ is called the infinitesimal generator of $T(\cdot)$. $T(\cdot)$ is of class $(C_0)$ if $T(t)$ converges strongly to $I$ as $t \to 0^+$. In this case, $T(\cdot)$ is also of class $(0, A)$ and $A^0$ is closed (see [2, p. 347]).

Let $S(t): x \to \int_0^t T(s)x \, ds (x \in X)$. The operators $C(t) := t^{-1}S(t), t > 0$, are the Cesàro averages of $T(\cdot)$, and the operators $A(\lambda) := \lambda R(\lambda), \lambda > \sigma_a$, are the Abel averages of $T(\cdot)$. Uniform ergodic theorems are concerned with the uniform operator convergence of $C(t)$ as $t \to \infty$ and of $A(\lambda)$ as $\lambda \to 0^+$. The result of Hille and Phillips [2, Theorem 18.8.4] deals with the uniform Abel-ergodicity of semigroups of class $(A)$ (a class slightly larger than $(0, A)$) under the assumption "$w_0 \leq 0$". The theorem of Lin [3] treats uniform Cesàro-ergodicity and Abel-ergodicity for semigroups of class $(C_0)$ under the assumption "$\lim_{t \to \infty} \|T(t)\|/t = 0$", which also implies $w_0 \leq 0$. However, as will be shown by an example in §3, it is possible for a semigroup to be uniformly ergodic while $-\infty = w = \sigma = \sigma_a < 0 < w_0$. Thus these well-known theorems do not apply universally.

The main purpose of this paper is to establish in §3 a uniform ergodic theorem (Theorem 4) for a general locally integrable semigroup, assuming the weaker condition "$\sigma_a \leq 0$". Under this assumption, the condition "$\lim_{t \to \infty} \|T(t)R(1)\|/t = 0$" (but not $\lim_{t \to \infty} \|T(t)\|/t = 0$) is necessary for uniform Cesàro-ergodicity of $T(\cdot)$. This is unlike the discrete case, where $\|T^n\|/n \to 0$ is necessary for the uniform ergodicity of $\{T^n\}$. When $T(\cdot)$ is a semigroup of class $(0, A)$, the uniform
Cesàro-ergodicity can be characterized precisely (Theorem 6). We shall begin with a uniform ergodic theorem (Theorem 1) for a general pseudo-resolvent. It will apply in §3 to provide conditions for uniform Abel-ergodicity of $T(\cdot)$.

2. Uniform ergodic theorems for pseudo-resolvents. In this section $R(\cdot)$ is a general pseudo-resolvent on an open subset $\Omega$ of the complex plane $\mathbb{C}$.

**Theorem 1.** Suppose that $0 \in \overline{\Omega}$. Then the following two statements are equivalent:

1. There is a $P \in B(X)$ such that $\|\lambda R(\lambda) - P\| \to 0$ as $\lambda \to 0$, $\lambda \in \Omega$.

2. $\|\lambda^2 R(\lambda)\| \to 0$ as $\lambda \to 0$, and the range $R(\lambda R(\lambda) - I)$ of $\lambda R(\lambda) - I$ is closed for some (and hence all) $\lambda \in \Omega$.

**Proof.** Note first that the range $R(\lambda R(\lambda) - I)$ and the null space $N(\lambda R(\lambda) - I)$ of $\lambda R(\lambda) - I$ are independent of $\lambda$ (see [6, p. 215]).

(1) $\Rightarrow$ (2). It is known from the mean ergodic theorem [6, p. 217] that $P$ is the projection onto $N(\lambda R(\lambda) - I)$ along $R(\lambda R(\lambda) - I)$. The fact that $\|\lambda R(\lambda)\| \to 0$ as $\lambda \to 0$ implies that $(\lambda R(\lambda) - I)|N(\lambda)\|$ is invertible for small $\lambda$ so that we have $R(\lambda R(\lambda) - I) \supset N(\lambda) = R(\lambda R(\lambda) - I)$, i.e., $R(\lambda R(\lambda) - I)$ is closed.

(2) $\Rightarrow$ (1). Fix a $\mu \neq 0$. Since $\mu R(\mu) - I$ has closed range, there exists a $M > 0$ such that each $y$ in $R(\mu R(\mu) - I)$ can be written as $y = (\mu R(\mu) - I)x$ for some $x$ satisfying $\|x\| \leq M\|y\|$. Using the resolvent equation we have

$$
\|\lambda R(\lambda) y\| = \|\lambda R(\lambda) (\mu R(\mu) - I)x\| = \left\| (\mu - \lambda)^{-1} [\lambda^2 R(\lambda) - \lambda \mu R(\mu)] x \right\| \leq |\mu - \lambda|^{-1} \left[ \|\lambda^2 R(\lambda)\| + \|\mu R(\mu)\| \right] M\|y\|,
$$

which implies that $\|\lambda R(\lambda)|R(\lambda R(\lambda) - I)|| \to 0$ as $\lambda \to 0$. Hence for small $\lambda$ the operator $K := (\lambda R(\lambda) - I)|R(\lambda R(\lambda) - I)$ is invertible and so we have that $R(\lambda R(\lambda) - I) = R(K) = R((\lambda R(\lambda) - I)^2)$. From this one easily deduces that $X = N(\lambda R(\lambda) - I) + R(\lambda R(\lambda) - I)$.

To show that the summation is direct, let $y$ be in $N(\lambda R(\lambda) - I) \cap R(\lambda R(\lambda) - I)$. Then $y = \lambda R(\lambda)y$ for all $\lambda$ in $\Omega$ and, as shown above, $\lambda R(\lambda)y \to 0$ as $\lambda \to 0$. Hence $y = 0$ and so $X = N(\lambda R(\lambda) - I) \oplus R(\lambda R(\lambda) - I)$. Let $P$ be the projection onto $N(\lambda R(\lambda) - I)$ along $R(\lambda R(\lambda) - I)$. Clearly we have $\|\lambda R(\lambda) - P\| = 0 \oplus [\lambda R(\lambda)|R(\lambda R(\lambda) - I)|| = 0$ as $\lambda \to 0$.

**Corollary 2.** Suppose that the domain $\Omega$ of $R(\cdot)$ is unbounded. Then the following statements are equivalent:

1. There is a $Q \in B(X)$ such that $\|\lambda R(\lambda) - Q\| \to 0$ as $|\lambda| \to \infty$, $\lambda \in \Omega$.

2. $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$, and $R(R(\lambda))$ is closed for some (and hence all) $\lambda \in \Omega$.

3. $R(\lambda) = Q(\lambda I - A)^{-1}$ for some $Q$, $A \in B(X)$ satisfying $Q^2 = Q$, $AQ = QA = A$.

**Proof.** Define $R_1(\lambda) := 1/\lambda - (1/\lambda^2)R(1/\lambda)$ for $\lambda \in \Omega_1 := \{\lambda \in C; 1/\lambda \in \Omega\}$. An easy computation shows that $R_1(\cdot)$ is a pseudo-resolvent on $\Omega_1$, which has the limit point $0$. Therefore, Theorem 1 applies to $R_1(\cdot)$, and the equivalence of (1) and
(2) follows by the facts that $\|\lambda^2 R_1(\lambda)\| \to 0$ as $\lambda \to 0$, $\lambda \in \Omega$, if and only if $R(\lambda) \to 0$ as $|\lambda| \to \infty$, $\lambda \in \Omega$, and that $\|\lambda R_1(\lambda) - P\| \to 0$ as $\lambda \to 0$ if and only if $\|\lambda R(\lambda) - Q\| \to 0$ as $|\lambda| \to \infty$, where $Q = I - P$. “(3) $\Rightarrow$ (1)” is obvious, and “(1) $\Rightarrow$ (3)” is proved in Theorem 18.8.2 of [2].

**Remark.** The operators $P$ and $Q$ turn out to be the linear projections with $R(P) = N(\lambda R(\lambda) - I)$, $N(P) = R(\lambda R(\lambda) - I) \cap N(Q) = R(\lambda)$ and $N(Q) = N(R(\lambda))$ for all $\lambda \in \Omega$.

3. Uniform ergodic theorems for locally integrable semigroups. All well-known uniform ergodic theorems for semigroups have been formulated for those of type $w_0 < 0$. We shall first give an example of a uniformly ergodic semigroup of class $(C_0)$ which satisfies $-\infty = a_0 < 0 < w_0$, and then prove a uniform ergodic theorem for general locally integrable semigroups under the assumption $a_0 \leq 0$.

**Example.** Let $1 < p < q < \infty$, and let $X$ be the set of all Lebesgue measurable functions $f$ on $(0, \infty)$ such that

$$11 \cdot \| f \| := \left( \int_0^\infty e^{p(\lambda)(s)} \| f(s) \|^p ds \right)^{1/p} + \left( \int_0^\infty | f(s) |^q ds \right)^{1/q} < \infty. $$

Then $(X, \| \cdot \|)$ is a Banach lattice which is reflexive whenever $p > 1$. For $\alpha > 0$ let $T_\alpha(t)$ be the semigroup defined by $(T_\alpha(t)f)(s) := e^{\alpha t} f(t + s)$ ($f \in X, s, t > 0$). Then $T_\alpha(t) = e^{\alpha(t/T_0(t))}$. It was shown in [1] that $\|T_\alpha(t)\| = 1$ for all $t > 0$ and for $T_0(\cdot)$ $w = a_0 = \sigma = -\infty$. Hence $T_\alpha(\cdot)$ has type $w_0 = \alpha$ and $w = a_0 = \sigma = -\infty$ for all $\alpha > 0$. Thus the infinitesimal generator $A_\alpha = d/ds + \alpha I$ of $T_\alpha(\cdot)$ has empty spectrum, and so

$$\| A(\lambda) \| = \left\| \lambda (\lambda I - A_\alpha)^{-1} \right\| \to 0 \cdot \| A_\alpha^{-1} \| = 0 \quad \text{as } \lambda \to 0. $$

Since $T_\alpha(t)$ are positive operators, we have for any nonnegative function $f$ in $X$

$$C(t)f = e^{-q(t)} \int_0^t T_\alpha(s)f ds \leq e^{-q(t)} \int_0^t e^{-s/T_\alpha(s)} T_\alpha(s) f ds \leq e^{-q(t)} \int_0^\infty e^{-s/T_\alpha(s)} T_\alpha(s) f ds = e^{q(t)} R(t^{-1}) f = e^{A(t^{-1})} f. $$

It follows that $\|C(t)\| \leq e\|A(t^{-1})\| \to 0$ as $t \to \infty$ (cf. [4, pp. 81 and 230]). Hence $T_\alpha(\cdot)$ is uniformly Abelian and Cesàro ergodic to 0.

We shall need the following lemma in the proof of Theorem 4:

**Lemma 3.** Let $T(\cdot)$ be a locally integrable semigroup, and let $S(t)$ and $R_s(\lambda)$ be as defined in §1. The equality $(T(t) - I)R_s(\lambda) = S(t)(\lambda R_s(\lambda) - I)$ holds for all $t > 0$ and $\Re \lambda > \sigma$.

**Proof.** Integration by parts gives that

$$R_s(\lambda)x = \int_0^\infty e^{-\lambda u} T(u)x du = \lambda \int_0^\infty e^{-\lambda u} S(u)x du. $$

Then we use the identity $(T(t) - I)S(u) = S(t)(T(u) - I)$ (see [5]) to obtain that

$$(T(t) - I)R_s(\lambda)x = \lambda \int_0^\infty e^{-\lambda u}(T(t) - I)S(u)x du
= \lambda \int_0^\infty e^{-\lambda u} S(t)(T(u) - I)x du = S(t)(\lambda R_s(\lambda) - I)x. $$

Theorem 4. Let $T(\cdot)$ be a locally integrable semigroup. Assume that $\sigma_a \leq 0$. Then the following statements are equivalent:

1. $T(\cdot)$ is uniformly Cesàro-ergodic.
2. $\|T(t)R(1)\|/t \to 0$ as $t \to \infty$, and $T(\cdot)$ is uniformly Abel-ergodic.
3. $\|T(t)R(1)\|/t \to 0$ as $t \to \infty$, and $R(R(1) - I)$ is closed.

Proof. (1) $\Rightarrow$ (2). We have for each $x \in X$ and $a > 0$,

$$
\|A(\lambda) - P\| x = \left\| \lambda^2 \int_0^\infty e^{-\lambda t}(S(t) - tP) x dt \right\|
\leq \left\{ \lambda^2 \int_0^a e^{-\lambda t} \left( \|S(t)\| + \|tP\| \right) dt + \lambda^2 a \|C(t) - P\| \right\} x
\leq \left\{ \sup_{t \leq a} \|S(t)\| + a \|P\| \lambda^2 a + \sup_{t > a} \|C(t) - P\| \right\} x.
$$

If $\|C(t) - P\| \to 0$ as $t \to \infty$, then it is easy to see from the above estimate that $\|A(\lambda) - P\| \to 0$ as $\lambda \to 0^+$. Then the fact that $N(P) = R(R(1) - I)$ (see the remark in §2) and Lemma 3 imply

$$
\|(T(t) - I)R(1)\|/t = \|C(t)(R(1) - I)\| = \|(C(t) - P)(R(1) - I)\|
\leq \|C(t) - P\| \|R(1) - I\| \to 0 \quad (t \to \infty).
$$

Hence the statement (2) holds when (1) holds.

"(2) $\Rightarrow$ (3)" is contained in Theorem 1.

(3) $\Rightarrow$ (1). First we prove that $\lim_{t \to \infty} \|T(t)R(1)\|/t = 0$ implies $\lim_{\lambda \to 0^+} \|\lambda^2 R(\lambda)\| = 0$. Given $\varepsilon > 0$, let $a > 0$ be such that $\|T(t)R(1)\| \leq \varepsilon t$ for all $t > a$. Using the resolvent equation we have for every $x \in X$

$$
\|\lambda^2 R(\lambda) x\| = \|\lambda^2 [R(1) + (1 - \lambda) R(\lambda) R(1)] x\|
\leq \lambda^2 \|R(1)\| \|x\| + |1 - \lambda| \lambda^2 \int_0^\infty e^{-\lambda t} \|T(t)R(1)\| x \| dt
\leq \lambda^2 \|R(1)\| \|x\| + |1 - \lambda| \left\{ \lambda^2 \|T(t)R(1)\| x \| dt + \varepsilon \lambda^2 \int_a^\infty e^{-\lambda t} dt \|x\| \right\}
\leq \left\{ \lambda^2 \|R(1)\| + |1 - \lambda| \left\{ \lambda^2 \|W(a)\| \|R(1)\| + \varepsilon \right\} \right\} \|x\|,
$$

where $W(a)$ denotes the operator from $X$ to $L_1(X,[0,a])$ defined by $W(a) x = T(\cdot) x$, which is known to be bounded (cf. [2, p. 58]). It is easily seen from the above estimate that $\|\lambda^2 R(\lambda)\| \to 0$ as $\lambda \to 0^+$.

Now the statement (2) of Theorem 1 holds, and it was proved there that

$$
X = N(R(1) - I) \oplus R(R(1) - I).
$$

Let $K$ be the restriction of $R(1) - I$ to $R(R(1) - I)$. Then $K$ is one-to-one, onto, and hence invertible. For $x \in R(R(1) - I)$ let $y = K^{-1} x$. By Lemma 3 we have

$$
\|C(t) x\| = \|C(t)(R(1) - I) y\| = \|t^{-1}(T(t) - I) R(1) K^{-1} x\|
\leq t^{-1} \left( \|T(t)R(1)\| + \|R(1)\| \right) \|K^{-1}\| \|x\|.
$$
This shows that \( \|C(t)\|N(R(1) - I) \rightarrow 0 \) as \( t \rightarrow \infty \). Let \( P \) be the projection onto \( R(R(1) - I) \) along \( R(R(1) - I) \). In order to prove that \( \|C(t) - P\| \) tends to 0 as \( t \rightarrow \infty \), it remains to show that the restriction of \( C(t) \) to \( N(R(1) - I) \) is an identity map for all \( t > 0 \). It suffices to show that \( N(R(1) - I) \subset N(T(t) - I) \). Let \( x \in N(R(1) - I) \). Then \( x \in N(\lambda R(\lambda) - I) \) for all \( \lambda > \sigma_a \) [6, p. 215], so that

\[
(T(t) - I)x = \lambda(T(t) - I)R(\lambda)x
\]

\[
= \lambda S(t)(\lambda R(\lambda) - I)x = S(t)(\lambda^2 R(\lambda) - \lambda)x,
\]

which converges to 0 as \( \lambda \rightarrow 0^+ \). Hence \( x \) belongs to \( N(T(t) - I) \). The proof is now completed.

**Corollary 5.** Let \( T(\cdot) \) be a locally integrable semigroup satisfying \( \sigma_a \leq 0 \) and \( \|T(t)R(1)\|/t \rightarrow 0 \) (\( t \rightarrow \infty \)). Then the following statements are equivalent:

1. \( T(\cdot) \) is uniformly Cesàro-ergodic.
2. \( T(\cdot) \) is uniformly Abel-ergodic.
3. \( R(R(1) - I) \) is closed.

**Remarks.**

1. If \( T(\cdot) \) is of class \((0, A)\) with generator \( A \), one has that \( R(R(1) - I) = R((I - A)^{-1} - I) = R(A(I - A)^{-1}) = R(A) \). Thus the theorem of Lin [3] is a specialization of Corollary 5.

2. It follows from Theorem 4 that the semigroup \( T_a(\cdot) \) in the previous example satisfies \( \|T_a(t)R(1)\|/t \rightarrow 0 \) (\( t \rightarrow \infty \)), while \( \|T_a(t)\|/t = e^{\alpha t} \rightarrow \infty \) in case \( \alpha > 0 \). Therefore, the hypothesis in Corollary 5 is in general strictly weaker than Lin’s \((\|T(t)\|/t \rightarrow 0)\), and it cannot be further weakened.

The next theorem gives a precise characterization for the uniform Cesàro-ergodicity of \((0, A)\) semigroups.

**Theorem 6.** Let \( T(\cdot) \) be a semigroup of class \((0, A)\). Then \( T(\cdot) \) is uniformly Cesàro ergodic if and only if \( i) \sigma \leq 0 \), \( ii) R(A) \) is closed, and \( iii) \|T(t)R(1)\|/t \rightarrow 0 \) as \( t \rightarrow \infty \).

This theorem is deduced from Theorem 4 and the following two propositions.

**Proposition 7.** If \( T(\cdot) \) is a semigroup of class \((0, A)\), then \( \sigma = \sigma_a \).

**Proof.** Since \( \sigma \leq \sigma_a \), it suffices to show that if \( \text{Re}\lambda > \sigma \), then \( \lambda I - A \) is invertible and \( R_\lambda(\lambda) = (\lambda I - A)^{-1} \).

We first prove that \( (\lambda I - A^0)R_\lambda(\lambda)x = x \) for \( x \in D(A^0) \). Let \( A_h = h^{-1}(T(h) - I) \).

Then we have

\[
A_h R_\lambda(\lambda)x = A_h \lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} T(s)x \, ds
\]

\[
= h^{-1} \lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} [T(s + h)x - T(s)x] \, ds
\]

\[
= h^{-1} (e^{\lambda h} - 1) R_\lambda(\lambda)x - h^{-1} \int_0^h e^{\lambda (h-s)} T(s)x \, ds,
\]
which tends to $\lambda R_s(\lambda)x - x$ as $h \to 0$ if $x \in D(A^0)$, by the continuity of $T(t)x$ at $t = 0$. Hence $R_s(\lambda)x \in D(A^0)$ and $A^0R_s(\lambda)x = \lambda R_s(\lambda)x - x$.

If $y \in X$, we can obtain a sequence $y_n \in D(A^0)$ such that $y_n \to y$. Then we have that $R_s(\lambda)y_n \to R_s(\lambda)y$ and $A^0R_s(\lambda)y_n - y_n = \lambda R_s(\lambda)y_n - y_n \to \lambda R_s(\lambda)y - y$. Since $A$ is the closure of $A^0$, we have proved that $R_s(\lambda)X \subset D(A)$ and $(\lambda I - A)R_s(\lambda) = I$.

It remains to show that $AR_s(\lambda)x = R_s(\lambda)Ax$ for $x \in D(A)$. Given $x \in D(A)$ there exists a sequence $x_n \in D(A^0)$ such that $x_n \to x$ and $A^0x_n \to Ax$. Since $A_h R_s(\lambda)x_n = R_s(\lambda)A_h x_n$, by letting $h \to 0$ we obtain that $A^0R_s(\lambda)x_n \to R_s(\lambda)Ax$ as $n \to \infty$. This and the fact that $R_s(\lambda)x_n \to R_s(\lambda)x$ show that $R_s(\lambda)x \in D(A)$ and $AR_s(\lambda)x = R_s(\lambda)Ax$.

**PROPOSITION 8.** If a locally integrable semigroup $T(\cdot)$ is strongly Cesàro-ergodic, then $\sigma \leq 0$, i.e. the Laplace transform $R_s(\lambda)$ exists for all $\lambda$ with $\Re \lambda > 0$.

**PROOF.** The uniform boundedness principle implies that $\|C(t)\| \leq M$ for all $t > 1$. Let $\Re \lambda > 0$. We have for all $v > u > 0$ and $x \in X$

\[
\left\| \int_u^v e^{-\lambda t} T(t) x \, dt \right\| \leq \left\| e^{-\lambda t} S(t) x \right\|_v + \lambda \int_u^v \left| e^{-\lambda t} S(t) x \right| \, dt \\
\leq \left( \left| e^{-\lambda v} |v| + |e^{-\lambda u}| |u| + |\lambda| \int_u^v e^{-\lambda t} dt \right) \|x\|,
\]

which shows that $\| \int_u^v e^{-\lambda t} T(t) \, dt \| \to 0$ as $u \to \infty$. Hence

\[ R_s(\lambda) = \lim_{t \to \infty} \int_0^t e^{-\lambda t} T(t) \, dt \]

exists.

**ACKNOWLEDGMENT.** The author wishes to thank the referee for his valuable suggestion.

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, NATIONAL CENTRAL UNIVERSITY, CHUNG - LI, TAIWAN 320, REPUBLIC OF CHINA**