THE EIGENFUNCTIONS OF COMPACT WEIGHTED ENDOMORPHISMS OF $C(X)$

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ABSTRACT. In this note we characterize the eigenmanifolds of compact operators $uC\Phi: f \mapsto u \cdot f \circ \Phi$ on $C(X)$ and determine their ascents. As an application we show an easy method for computing the eigenmanifolds of a matrix with at most one nonzero element in each row.

In the sequel $X$ will always denote a compact Hausdorff space, $u$ a function in $C(X)$, and $\Phi$ a continuous function from $X$ to $X$. Let $\Phi_n$ be the $n$th iterate of $\Phi$; i.e., $\Phi_0(x) = x$ and $\Phi_n(x) = \Phi(\Phi_{n-1}(x))$ for $n > 0$ and $x \in X$. $c \in X$ is called a fixed point of $\Phi$ of order $n$ if $n$ is a positive integer, $\Phi_n(c) = c$, and $\Phi_k(c) \neq c$ for $k = 1, \ldots, n - 1$.

By $uC\Phi$ we denote the operator $uC\Phi: f \mapsto u \cdot f \circ \Phi$ on $C(X)$. This is a weighted endomorphism, and every weighted endomorphism may be represented in this way (see Kamowitz [1]). Kamowitz [1] proved the following result:

**Theorem A.** Suppose $X$ is a compact Hausdorff space, $u$ in $C(X)$, and $\Phi$ a continuous function from $X$ into $X$.

1. The map $uC\Phi: f \mapsto u \cdot f \circ \Phi$ is compact iff for each connected component $C$ of $\{x | u(x) \neq 0\}$ there exists an open set $V \supset C$ such that $\Phi$ is constant on $V$.

2. If $uC\Phi$ is compact, then $\sigma(uC\Phi) \setminus \{0\} = \{\lambda | \lambda^n = u(c) \cdots u(\Phi_{n-1}(c)) \text{ for some positive integer } n \text{ and some fixed point } c \text{ of } \Phi \text{ of order } n, \lambda \neq 0\}$.

Our aim here is to characterize the eigenfunctions of a compact $uC\Phi$. To do that we need some more notation: We always assume that $\Phi$ satisfies the conditions of Theorem A(1) so that $uC\Phi$ is compact. We call $x, y \in X$ equivalent ($x \sim y$) if there exist $n, m \in \{0, 1, 2, \ldots\}$ so that $\Phi_n(x) = y$ and $\Phi_m(y) = x$. The equivalence classes are denoted by $[x]$. For any $\lambda$ in $C \setminus \{0\}$ let $C_\lambda := \{c \in X | c$ is a fixed point of $\Phi$ of order $n$ for some positive integer $n$ and $\lambda^n = u(c) \cdots u(\Phi_{n-1}(c))\}$. Obviously if $x \sim y$ and $x \in C_\lambda$, then $y$ is in $C_\lambda$, so let $\hat{C}_\lambda := \{[x] | x \in C_\lambda\}$ and $m_\lambda$ be the number of equivalent classes in $\hat{C}_\lambda$. $m_\lambda$ is finite by Theorem B and the compactness of $uC\Phi$. For every $c \in C_\lambda$ let $h_{c,\lambda}$ denote the following function from $X$ to $C$ or $R$ respectively:

$$h_{c,\lambda}(x) := \begin{cases} \lambda^{-r}u(x) \cdots u(\Phi_r(x)) & \text{for every } r \text{ in } \{0, 1, 2, \ldots\} \text{ and } x \in \Phi_r^{-1}(\{c\}), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $h_{c,\lambda}$ is well defined (remember that e.g. $c$ is in every $\Phi_k^{-1}(\{c\})$ if $c$ is a fixed point of $\Phi$ of order $n$, but then $\lambda^{kn} = u(c) \cdots u(\Phi_{kn-1}(c))$). Furthermore
\{h_{c_1, \lambda}, \ldots, h_{c_m, \lambda}\} is linearly dependent iff, for some \(i \neq j\), \(c_i \sim c_j\). Finally, let \(W_0 := W := \{x|u(x) \neq 0\}\) and \(W_k := \Phi(W \cap W_{k-1})\) for \(k > 0\). For additional notation see Taylor [2].

The principal result of this note is the following theorem.

**THEOREM B.** (1) Let \(\lambda \in \sigma(uC_\Phi) \setminus \{0\}\) and \(\{c_1, \ldots, c_m, \lambda\}\) be representative elements of all equivalence classes in \(C_\lambda\). Then \(\{h_{c_1, \lambda}, \ldots, h_{c_m, \lambda}\}\) is a basis for \(N(\lambda - uC_\Phi)\) and \(\alpha(\lambda - uC_\Phi) = 1\), where \(\alpha(\lambda - uC_\Phi)\) denotes the ascent of \(\lambda - uC_\Phi\).

(2) The case \(\lambda = 0\): If \(n > 0\), then \(N((uC_\Phi)^n) = \{f \in \mathcal{C}(X)|f(x) = 0\text{ for every } x \in W_n\}\).

Notice that (1) also states that the functions \(h_{c, \lambda}\) are continuous.

We will break up the proof by proving several propositions.

**PROPOSITION 1.** Let \(\lambda \in \sigma(uC_\Phi) \setminus \{0\}\). Then \(h_{c, \lambda}\) is an eigenfunction for \(\lambda\) for every \(c \in C_\lambda\); that is,

(i) \(\lambda h_{c, \lambda}(x) = u(x)h_{c, \lambda}(\Phi(x))\) for all \(x \in X\),

(ii) \(h_{c, \lambda}\) is continuous.

**PROOF.** (i) Let \(x \in X\). If \(x \in \Phi^{-r}(\{c\})\) for some \(r > 0\), then

\[
\lambda h_{c, \lambda}(x) = u(x)(\lambda^{-(r-1)}u(\Phi(x)) \cdots u(\Phi(x))) = u(x)h_{c, \lambda}(\Phi(x)).
\]

If \(x \notin \Phi^{-r}(\{c\})\) for every \(r > 0\), then the same is true for \(\Phi(x)\), so \(\lambda h_{c, \lambda}(x) = 0 = u(x)h_{c, \lambda}(\Phi(x))\).

(ii) (1) Since \(u\) is continuous, \(B = \{x|\ |u(x)| \geq |\lambda|\}\) is compact. As \(W\) may be covered with open sets \(V_\beta\), so that \(\Phi\) is constant on each \(V_\beta\), \(\Phi(B)\) is finite, of cardinality \(N\), say. Let \(x \in X\) such that \(h_{c, \lambda}(x) \neq 0\), and \(r\) the minimal number so that \(x \in \Phi^{-r}(\{c\})\). Now \(x, \Phi(x), \ldots, \Phi_r(x)\) are distinct, whence

\[
|h_{c, \lambda}(x)| = |u(x)/\lambda| \cdot |u(\Phi(x))/\lambda| \cdots |u(\Phi_{r-1}(x))/\lambda| \cdot |u(c)|
\]

\[
\leq \max\{1,(||u||_\infty/|\lambda|)^N\} \cdot |u(c)| =: M.
\]

Therefore \(h_{c, \lambda}\) is bounded on \(X\).

(2) Let \(x \in X\). If \(u(x) = 0\), then \(h_{c, \lambda}(x) = 0\) and for every \(\varepsilon > 0\) there is a neighborhood \(U\) of \(x\) so that \(|u(y)| < \varepsilon|\lambda|/M\) for every \(y \in U\). Therefore

\[
|h_{c, \lambda}(y)| = |\lambda|^{-1}|h_{c, \lambda}(\Phi(x))| |u(y)| < \varepsilon
\]

for every \(y \in U\) and thus \(h_{c, \lambda}\) is continuous at \(x\). If \(u(x) \neq 0\), then \(\Phi\) is constant on an open neighborhood \(U\) of \(x\) and therefore

\[
|h_{c, \lambda}(x) - h_{c, \lambda}(y)| = |\lambda|^{-1}|h_{c, \lambda}(\Phi(x))| |u(x) - u(y)| < \varepsilon
\]

for a suitable neighborhood \(U' \subset U\) of \(x\) and every \(y \in U'\). So \(h_{c, \lambda}\) is continuous.

**PROPOSITION 2.** Let \(\lambda \in \sigma(uC_\Phi), \lambda \neq 0\), and \(f\) an eigenfunction for \(\lambda\). Then

(i) For every \(c \in C_\lambda\) there exists \(\alpha(c)\) such that \(f(x) = \alpha(c)h_{c, \lambda}(x)\) for every \(r \geq 0\) and \(x \in \Phi^{-r}(\{c\})\).

(ii) If \(f \notin \Phi^{-1}(\{c\})\) for every \(c \in C_\lambda\) and \(r \geq 0\), then \(f(x) = 0\).

**PROOF.** (i) Let \(c \in C_\lambda\) and \(\alpha(c) := f(c)/u(c)\) (remember \(\lambda \neq 0!\)). Then for \(r \geq 0\) and \(x \in \Phi^{-r}(\{c\})\) we have by iteration

\[
f(x) = \lambda^{-r}u(x)u(\Phi(x)) \cdots u(\Phi_{r-1}(x))f(\Phi_r(x)) = \alpha(c)h_{c, \lambda}(x).
\]
(ii) This part of the proof is actually the same as for Proposition 4 in [1] and is repeated here for the sake of completeness:

Let \( x \in \Phi^{-1}_r(\{c\}) \) for every \( c \in C_\lambda, \ r \geq 0. \) If \( x \) is a fixed point of \( \Phi \), of order \( n \), say, then by iteration \( f(x) = \lambda^{-n}u(x) \cdots u(\Phi_{n-1}(x))f(x) \) and, since \( x \in C_\lambda \), we conclude that \( f(x) = 0. \)

If \( x \in \Phi^{-1}_r(\{c\}) \) for some fixed point \( c \notin C_\lambda \) and \( r \geq 1 \), then, since \( f(c) = 0 \), we have \( f(x) = \lambda^{-n}u(x) \cdots u(\Phi_{r-1}(x))f(c) = 0. \)

Finally, we may suppose that all \( \Phi_r(x) \) are distinct. Let \( \delta := |\lambda|/2. \) Since \( B := \{x \ | \ |u(x)| \geq \delta \} \) is compact and by Theorem A \( W \) may be covered by open sets on which \( \Phi \) is constant, \( \Phi(B) \) is finite, of cardinality \( N \), say. Therefore for every \( n > N \)

\[
|f(x)| = |u(x)/\lambda||u(\Phi(x))/\lambda| \cdots |u(\Phi_{n-1}(x))/\lambda||f(\Phi_n(x))|
\]\[
\leq \left( ||u||_\infty/|\lambda| \right)^N 2^{N-n}||f||_\infty \to 0 \quad (n \to \infty).
\]

Thus \( f(x) = 0. \) Q.E.D.

Let \( \{c_1, \ldots, c_m\} \) be representative elements of all equivalence classes in \( \tilde{C}_\lambda. \)

Then \( \{h_{c_1, \lambda}, \ldots, h_{c_m, \lambda}\} \) is a basis for \( \mathcal{N}(\lambda - A) \) if \( 0 \neq \lambda \in \sigma(uC_\Phi). \) So what remains to be done for part (1) of Theorem B is

**PROPOSITION 3.** Let \( 0 \neq \lambda \in \sigma(uC_\Phi) \) and \( f \in \mathcal{N}((\lambda - uC_\Phi)^2). \) Then \( f \in \mathcal{N}(\lambda - uC_\Phi). \)

**PROOF.** Since \( g := (\lambda - uC_\Phi)f \) is an eigenfunction for \( \lambda \), we know by Proposition 2 that if \( x \) is not in \( \Phi^{-1}_r(\{c\}) \) for some \( c \in C_\lambda \) and \( r \geq 0 \), then \( g(x) = 0. \) If \( c \in C_\lambda \) there exists \( \alpha(c) \) so that \( g(x) = \alpha(c)h_{c, \lambda}(x) \) for every \( r \geq 0 \) and \( x \in \Phi^{-1}_r(\{c\}) \) by Proposition 2, so we have to show that \( \alpha(c) = 0. \) Let \( c \) be of order \( n. \) Since by iteration

\[
f = n \cdot g + (uC_\Phi)^n f / \lambda^n,
\]
evaluation at \( c \) yields

\[
f(c) = n\alpha(c)h_{c, \lambda}(c)/\lambda + f(c),
\]
for \( g \) is an eigenfunction and \( \lambda^n = u(c) \cdots u(\Phi_n(c)). \) Therefore \( \alpha(c) = 0. \)

So far we have proved Theorem B(1). Part (2) follows from

**PROPOSITION 4.** \( (uC_\Phi)^kf = 0 \iff f(x) = 0 \) for every \( x \in W_k. \)

**PROOF.** By induction:

(\( \Rightarrow \)) Let \( k = 1 \) and \( uC_\Phi f = 0. \) Then for any \( x \in W \) we have \( 0 = u(x)f(\Phi(x)), \) whence \( f(\Phi(x)) = 0. \) If \( k > 1 \) and \( (uC_\Phi)^kf = 0, \) we know by induction that \( u(x)f(\Phi(x)) = 0 \) for every \( x \in W_{k-1}. \) Furthermore, if \( x \in W \), then \( u(x) \neq 0, \) so that \( f(\Phi(x)) = 0. \) Thus \( f \) vanishes on \( W_k. \)

(\( \Leftarrow \)) Let \( k = 1 \) and \( f(x) = 0 \) for every \( x \in W_1. \) For \( x \in X \) either \( x \in W \) and therefore \( f(\Phi(x)) = 0 \) or \( x \notin W \) and \( u(x) = 0. \) Thus \( uC_\Phi f = 0. \) Now let \( k > 1 \) and \( f(x) = 0 \) for every \( x \in W_k. \) We have to show that \( u(x)f(\Phi(x)) = 0 \) for every \( x \in W_{k-1}. \), because then the assertion follows by induction hypothesis. But this is trivial since either \( x \notin W \) and \( u(x) = 0, \) or \( \Phi(x) \in W_k \) and \( f(\Phi(x)) = 0, \) if \( x \in W_{k-1}. \)

**EXAMPLE 1.** We want to give an example for Theorem B(2) that the case \( \mathcal{N}((uC_\Phi)^n) \neq \mathcal{N}((uC_\Phi)^{n+1}) \) for ever \( n \) may occur. Let \( X := \{0\} \cup \{1/n | n \in \mathbb{N}\} \) with the topology induced by the usual topology on \( \mathbb{R} \) so that \( X \) is compact. Let
$u(x) = x$ and $\Phi(1/n) = 1/(n + 1)$, $\Phi(0) = 0$. These are continuous functions satisfying the conditions of Theorem A. Therefore $uC_\Phi$ is a compact operator on $C(X)$, where $C(X)$ may obviously be identified with $c(\mathbb{N}) := \{(a_n)_{n \in \mathbb{N}} | \lim_{n \to -\infty} a_n$ exists$\}$. Since there are no fixed points $c \neq 0$ of $\Phi$ of any order, $\sigma(uC_\Phi) = \{0\}$ by Theorem A. Now $W_k = \{x \in X | 0 < x < 1/k\}$, so $\mathcal{N}((uC_\Phi)^k) = \{(a_n) | a_n = 0$ for every $n > k\}$ and the union of all $\mathcal{N}((uC_\Phi)^k)$ is exactly the set of all $(a_n)$ satisfying $a_n = 0$ for all but finitely many $n$.

EXAMPLE 2. We give an application of our results to the finite-dimensional case. Let $X = \{1, \ldots, n\}$ with the discrete topology. Then $C(X)$ will be identified with $K^n$, where $K = \mathbb{C}$ or $K = \mathbb{R}$ is the underlying scalar field. Every linear operator may (and will) be identified with the matrix $(a_{ij})_{1 \leq i, j \leq n}$ with $a_{ij} = (A \delta_j)(i)$, where $\delta_j(i) = 1$, $\delta_j(i) = 0$ if $i \neq j$.

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$\mathcal{N}(A) = \{(x_n) | x_1 = x_2 = x_5 = x_6 = 0\}$, $\mathcal{N}(A^2) = \{(x_n) | x_5 = x_2 = 0\} = \mathcal{N}(A^3)$.

If $A = uC_\Phi$, then $a_{ij} = u(i)$ if $j = \Phi(i)$ and $a_{ij} = 0$ otherwise, so there is at most one nonzero element in each row. Conversely let $A$ have this property. Then for $i = 1, \ldots, n$ let $j = \Phi(i)$ and $u(i) = a_{ij}$, if $a_{ij}$ is the unique nonzero element in row $i$. If $a_{ij} = 0$ for all $j = 1, \ldots, n$ we let $i = \Phi(i)$ and $u(i) = 0$. Then obviously $A = uC_\Phi$.

Now the eigenvalues and eigenvectors are easily determined: first find out all cycles of $\Phi$, e.g. by drawing $n$ dots with numbers $1, \ldots, n$ and an arrow from dot $j$ to dot $i$ if $\Phi(i) = j$, adding $u(i)$ to that arrow for later purposes. For each cycle multiply all the $u(i)$ of this cycle and calculate the $k$th roots, where $k$ denotes the number of elements of this cycle: these are all eigenvalues possibly except 0.

Take one eigenvalue $\lambda \neq 0$ and a cycle corresponding to that $\lambda$. Choose an arbitrary dot $j$, say, of that cycle and set $x_j := u(j)$. Now follow the arrows. If you reach dot $i$ from dot $k$ let $x_i$ be the product of $\lambda^{-1}u_i$ and $x_k$. When you are done with all the dots which belong to the „connected component“ containing the cycle set all other $x_i = 0$. This is an eigenvector for $\lambda$.

If you do this for every cycle corresponding to $\lambda$ you get a basis for the eigenspace $\mathcal{N}(\lambda - A)$.

In order to determine $\mathcal{N}(A^r)$ remove all arrows where $u_i = 0$. Now $\mathcal{N}(A)$ consists of all $(x_k)$, where $x_k = 0$ if there is a directed path of length one starting in dot $k$ (to dot $k$ itself or any other dot), and $x_k$ is arbitrary otherwise. Similarly for $\mathcal{N}(A^r)$, $r > 1$: “one” has to be replaced by “$r$” and it is allowed to “use” the same arrow more than one time.
There is a diagonalization for $A$ iff $\mathcal{N}(A) = \mathcal{N}(A^2)$. Of course all these results are easily obtained by direct verification as well.

REFERENCES


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