ON A QUESTION OF N. SALINAS
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ABSTRACT. In [5], Salinas asked the following question: If $T = (T_1, \ldots, T_n)$ consists of commuting hyponormal operators, is it true that (1) $\delta(T - \lambda) = d(\lambda, \sigma(T))$ and (2) $r(T) = ||D_T||$? He proved that, for a doubly commuting $n$-tuple of quasinormal operators, (2) was true and (1) was true for $\lambda = 0$.

In this paper we shall show that (2) holds for a doubly commuting $n$-tuple of hyponormal operators and give an example of a subnormal operator which does not satisfy (1).

1. Introduction. Let $H$ be a complex Hilbert space, and all operators on $H$ will be assumed to be linear and bounded. $\mathcal{L}$ will denote the algebra of all operators on $H$.

Let $T = (T_1, \ldots, T_n) \in \mathcal{L}^n$. We shall say that a point $z = (z_1, \ldots, z_n)$ of $\mathbb{C}^n$ is in the joint approximate point spectrum $\sigma_a(T)$ of $T$ if there exists a sequence $\{x_k\}$ of unit vectors in $H$ such that

$$||(T_i - z_i)^*x_k|| \to 0 \text{ as } k \to \infty, \quad i = 1, \ldots, n.$$ 

A point $z = (z_1, \ldots, z_n)$ of $\mathbb{C}^n$ is said to be in the joint defect spectrum $\sigma_d(T)$ of $T$ if there exists a sequence $\{x_k\}$ of unit vectors in $H$ such that

$$||(T_i - z_i)^*x_k|| \to 0 \text{ as } k \to \infty, \quad i = 1, \ldots, n.$$ 

The joint numerical range of $T$ is the subset $W(T)$ of $\mathbb{C}^n$ such that

$$W(T) = \{((T_1x, x), \ldots, (T_nx, x)) : x \in H, ||x|| = 1\}.$$ 

The joint operator norm and the joint approximate spectral radius of $T$, denoted by $||D_T||$ and $r(T)$, respectively, are defined by

$$||D_T|| = \sup \left\{ \left( \sum_{i=1}^{n} ||T_i x||^2 \right)^{1/2} : x \in H, ||x|| = 1 \right\}$$

and

$$r(T) = \sup \left\{ \left( \sum_{i=1}^{n} |z_i|^2 \right)^{1/2} : (z_1, \ldots, z_n) \in \sigma(T) \right\},$$

respectively. Let

$$\delta(T - \lambda) = \inf \left\{ \left( \sum_{i=1}^{n} ||(T_i - \lambda_i)x||^2 \right)^{1/2} : x \in H, ||x|| = 1 \right\}.$$
Let $\sigma_T(T)$ be Taylor’s joint spectrum of $T$. We refer the reader to [7] for the definition of $\sigma_T(T)$.

In the sequel, we let $T^*T$ be the $n$-tuple $(T_1^*T_1, \ldots, T_n^*T_n)$.

2. Theorem.

**Theorem.** Let $T = (T_1, \ldots, T_n)$ be a doubly commuting $n$-tuple of hyponormal operators.

Then $\|DT\| = r_\pi(T)$.

We shall need the following three facts.

**Theorem A (Curto [2]).** Let $T = (T_1, \ldots, T_n)$ be a doubly commuting $n$-tuple of hyponormal operators.

Then $\sigma_T(T) = \sigma(T)$.

**Theorem B (Dash [3]).** Let $T = (T_1, \ldots, T_n)$ be a doubly commuting $n$-tuple of normal operators.

Then $W(T) = \sigma(T)$.

**Theorem C (Chō and Takaguchi [1]).** Let $T = (T_1, \ldots, T_n)$ be a doubly commuting $n$-tuple of hyponormal operators. If $r = (r_1, \ldots, r_n) \in \sigma_T(T^*T)$, then there exists a point $z = (z_1, \ldots, z_n) \in \sigma(T)$ such that $z_i = r_i (i = 1, \ldots, n)$.

**Proof of the Theorem.**

$$\|DT\|^2 = \sup \left\{ \sum_{i=1}^{n} \|T_i x\|^2 : \|x\| = 1 \right\} = \sup \left\{ \sum_{i=1}^{n} (T_i^*T_i x, x) : \|x\| = 1 \right\}.$$ 

So there exists $r = (r_1, \ldots, r_n)$ in $W(T^*T)$ such that $\|DT\|^2 = \sum_{i=1}^{n} r_i$. By Theorem B, it follows that $W(T^*T) = \sigma(T^*T)$. If there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ in $\sigma(T^*T)$ and $0 < t < 1$ such that $r = t\alpha + (1 - t)\beta$, then we have

$$\sum_{i=1}^{n} r_i < \max \left\{ \sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \beta_i \right\},$$

which is a contradiction to the choice of $r$. Therefore, $r = (r_1, \ldots, r_n)$ belongs to $\sigma(T^*T)$ and there exists $z = (z_1, \ldots, z_n) \in \sigma(T)$ such that $\sum_{i=1}^{n} r_i = |z|^2$ by Theorems C and A.

Let $\{x_k\}$ be a sequence of unit vectors in $H$ such that

$$\|(T_i - z_i)^* x_k\| \to 0 (k \to \infty), \quad i = 1, \ldots, n.$$ 

Hence,

$$\sum_{i=1}^{n} \|(T_i - z_i) x_k\|^2 \leq |z|^2 - 2 \sum_{i=1}^{n} \Re \bar{z}_i (T_i x_k, x_k) + |z|^2,$$

and, since $(T_i x_k, x_k) = (x_k, T_i^* x_k) = (x_k, (T_i - z_i)^* x_k) + z_i \to z_i (k \to \infty)$, $i = 1, \ldots, n$, we have

$$\sum_{i=1}^{n} \|(T_i - z_i) x_k\|^2 \to 0, \quad \text{as } k \to \infty.$$ 

Therefore, $z = (z_1, \ldots, z_n)$ belongs to $\sigma_T(T)$.

So the proof is complete.
3. Examples. Let $H$ be an infinite-dimensional Hilbert space with orthonormal basis $\{e_k\}_{k=1}^{\infty}$. Consider the weighted shift $T$ on $H$ such that
\[
Te_1 = \frac{1}{2} e_2, \quad Te_k = e_{k+1} \quad (k = 2, 3, \ldots).
\]
Then $T$ is subnormal (cf. p. 379 in Stampfli [6]), $\sigma_\pi(T) = \{z: |z| = 1\}$ (cf. Furuta [4]) and evidently $\delta(T) = \frac{1}{2}$. Therefore, $\delta(T) < 1 = d(0, \sigma_\pi(T))$, so that (1) is in general false for a subnormal $n$-tuple.

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REFERENCES

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