ON A QUESTION OF N. SALINAS

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ABSTRACT. In [5], Salinas asked the following question: If \( T = (T_1, \ldots, T_n) \) consists of commuting hyponormal operators, is it true that (1) \( \delta(T - \lambda) = d(\lambda, \sigma(T)) \) and (2) \( r_\pi(T) = ||D_T|| \)? He proved that, for a doubly commuting \( n \)-tuple of quasinormal operators, (2) was true and (1) was true for \( \lambda = 0 \). In this paper we shall show that (2) holds for a doubly commuting \( n \)-tuple of hyponormal operators and give an example of a subnormal operator which does not satisfy (1).

1. Introduction. Let \( H \) be a complex Hilbert space, and all operators on \( H \) will be assumed to be linear and bounded. \( \mathcal{L} \) will denote the algebra of all operators on \( H \).

Let \( T = (T_1, \ldots, T_n) \in \mathcal{L}^n \). We shall say that a point \( z = (z_1, \ldots, z_n) \) of \( \mathbb{C}^n \) is in the joint approximate point spectrum \( \sigma_\pi(T) \) of \( T \) if there exists a sequence \( \{x_k\} \) of unit vectors in \( H \) such that
\[
\|(T_i - z_i)x_k\| \to 0 \quad (k \to \infty), \quad i = 1, \ldots, n.
\]
A point \( z = (z_1, \ldots, z_n) \) of \( \mathbb{C}^n \) is said to be in the joint defect spectrum \( \sigma_\delta(T) \) of \( T \) if there exists a sequence \( \{x_k\} \) of unit vectors in \( H \) such that
\[
\|(T_i - z_i)x_k\| \to 0 \quad (k \to \infty), \quad i = 1, \ldots, n.
\]
The joint numerical range of \( T \) is the subset \( W(T) \) of \( \mathbb{C}^n \) such that
\[
W(T) = \{(\langle T_1x, x \rangle, \ldots, \langle T_nx, x \rangle) : x \in H, \|x\| = 1\}.
\]
The joint operator norm and the joint approximate spectral radius of \( T \), denoted by \( ||D_T|| \) and \( r_\pi(T) \), respectively, are defined by
\[
||D_T|| = \sup \left\{ \left( \sum_{i=1}^{n} \|T_ix\|^2 \right)^{1/2} : x \in H, \|x\| = 1 \right\}
\]
and
\[
r_\pi(T) = \sup \left\{ \left( \sum_{i=1}^{n} |z_i|^2 \right)^{1/2} : (z_1, \ldots, z_n) \in \sigma_\pi(T) \right\},
\]
respectively. Let
\[
\delta(T - \lambda) = \inf \left\{ \left( \sum_{i=1}^{n} \|(T_i - \lambda_i)x\|^2 \right)^{1/2} : x \in H, \|x\| = 1 \right\}.
\]
Let \( \sigma_T(T) \) be Taylor's joint spectrum of \( T \). We refer the reader to [7] for the definition of \( \sigma_T(T) \).

In the sequel, we let \( T^*T \) be the \( n \)-tuple \( (T_1^*T_1, \ldots, T_n^*T_n) \).

2. Theorem.

THEOREM. Let \( T = (T_1, \ldots, T_n) \) be a doubly commuting \( n \)-tuple of hyponormal operators.

Then \( \|D_T\| = \sigma_\pi(T) \).

We shall need the following three facts.

THEOREM A (CURTO [2]). Let \( T = (T_1, \ldots, T_n) \) be a doubly commuting \( n \)-tuple of hyponormal operators.

Then \( \text{ot}(T) = \sigma_\delta(T) \).

THEOREM B (DASH [3]). Let \( T = (T_1, \ldots, T_n) \) be a doubly commuting \( n \)-tuple of normal operators.

Then \( W(T) = \sigma_\Omega(T) \) (= convex hull of \( \sigma_T(T) \)).

THEOREM C (CHÔ AND TAKAGUCHI [1]). Let \( T = (T_1, \ldots, T_n) \) be a doubly commuting \( n \)-tuple of hyponormal operators. If \( \tau = (\tau_1, \ldots, \tau_n) \in \sigma_T(T^*T) \), then there exists a point \( z = (z_1, \ldots, z_n) \in \sigma_T(T) \) such that \( z_i = \sqrt{\tau_i} (i = 1, \ldots, n) \).

PROOF OF THE THEOREM.

\[ \|D_T\|^2 = \sup \left\{ \sum_{i=1}^{n} \|T_i x\|^2 : \|x\| = 1 \right\} = \sup \left\{ \sum_{i=1}^{n} (T_i^* T_i x, x) : \|x\| = 1 \right\} \]

So there exists \( r = (r_1, \ldots, r_n) \) in \( W(T^*T) \) such that \( \|D_T\|^2 = \sum_{i=1}^{n} r_i \). By Theorem B, it follows that \( W(T^*T) = \sigma_\Omega(T^*T) \). If there exist \( \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \) in \( \sigma_T(T^*T) \) and \( 0 < t < 1 \) such that \( r = t \alpha + (1-t) \beta \), then we have

\[ \sum_{i=1}^{n} r_i < \max \left\{ \sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \beta_i \right\}, \]

which is a contradiction to the choice of \( r \). Therefore, \( r = (r_1, \ldots, r_n) \) belongs to \( \sigma_T(T^*T) \) and there exists \( z = (z_1, \ldots, z_n) \) in \( \sigma_\delta(T) \) such that \( \sum_{i=1}^{n} r_i = |z|^2 \) by Theorems C and A.

Let \( \{x_k\} \) be a sequence of unit vectors in \( H \) such that

\[ \|(T_i - z_i)^* x_k\| \to 0 (k \to \infty), \quad i = 1, \ldots, n. \]

Hence,

\[ \sum_{i=1}^{n} \|(T_i - z_i) x_k\|^2 \leq |z|^2 - 2 \sum_{i=1}^{n} \text{Re} \bar{z}_i (T_i x_k, x_k) + |z|^2, \]

and, since \( (T_i x_k, x_k) = (x_k, T_i^* x_k) = (x_k, (T_i - z_i)^* x_k) + z_i \to z_i (k \to \infty), \ i = 1, \ldots, n \), we have

\[ \sum_{i=1}^{n} \|(T_i - z_i) x_k\|^2 \to 0, \quad \text{as } k \to \infty. \]

Therefore, \( z = (z_1, \ldots, z_n) \) belongs to \( \sigma_\pi(T) \).

So the proof is complete.
3. Examples. Let $H$ be an infinite-dimensional Hilbert space with orthonormal basis $\{e_k\}_{k=1}^{\infty}$. Consider the weighted shift $T$ on $H$ such that

$$Te_1 = \frac{1}{2} \cdot e_2, \quad Te_k = e_{k+1} \quad (k = 2, 3, \ldots).$$

Then $T$ is subnormal (cf. p. 379 in Stampfli [6]), $\sigma_\pi(T) = \{z: |z| = 1\}$ (cf. Furuta [4]) and evidently $\delta(T) = \frac{1}{2}$. Therefore, $\delta(T) < 1 = d(0, \sigma_\pi(T))$, so that (1) is in general false for a subnormal $n$-tuple.

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REFERENCES

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