A NOTE ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A HETEROGENEOUS NONLINEAR REACTION-DIFFUSION SYSTEM

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ABSTRACT. A result on asymptotic behavior of solutions to a heterogeneous nonlinear reaction-diffusion system with homogeneous Neumann boundary condition is obtained, which improves the results in [5].

1. Introduction. The large time behavior of solutions to systems of nonlinear reaction-diffusion equations, namely, the decay to the spatially homogeneous solutions, was studied by E. Conway, D. Hoff, and J. Smoller [2]. The corresponding heterogeneous case was considered by Y. Su. [5]. This paper improves the main result of [5].

We consider the heterogeneous system

(1.1) \[ u_t = (1/\varepsilon^2)D \Delta u + f(x,t,u), \quad (x,t) \in \Omega \times \mathbb{R}^+, \]

with initial condition

(1.2) \[ u(x,0) = u_0(x), \quad x \in \bar{\Omega}, \]

and homogeneous Neumann boundary condition

(1.3) \[ \frac{\partial u}{\partial n}(x,t) = 0, \quad (x,t) \in \partial \Omega \times \mathbb{R}^+, \]

where \( u \in \mathbb{R}^n, \Omega \subset \mathbb{R}^m \) is a bounded domain, \( \partial \Omega \) is the boundary of \( \Omega \), \( \partial / \partial n \) represents the outward normal derivative on \( \partial \Omega \), \( D \) is an \( n \times n \) matrix, \( f: \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n \) is a nonlinear vector function, and \( \varepsilon > 0 \) is a small parameter concerning the diffusion strength.

We will approximate the solutions of (1.1)--(1.3) by corresponding spatially homogeneous solutions and give the estimate of the "error".

2. Main result. We need the following assumptions:

(A-1) System (1.1) admits a bounded invariant region [1] \( \Sigma \subset \mathbb{R}^n \).

(A-2) The diffusion matrix \( D \) is positive definite, i.e.,

\[ \langle Du, v \rangle \geq d|v|^2 \quad \forall v \in \mathbb{R}^n, \]

where \( d > 0 \) is the smallest (positive) eigenvalue of \( D \).

(A-3) \( f(x,t,u) \in C^{1,0,1}(\Omega \times \mathbb{R}^+ \times \mathbb{R}^n) \), i.e., \( f \) has bounded first order derivatives with respect \( x \in \Omega \) and \( u \in \Sigma \).

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Denote
\[ M = \max\{|df_u|: u \in \Sigma, \ x \in \bar{\Omega}, \ t \in \mathbb{R}^+\}, \]
\[ N(t) = \max\{|df_x|: u \in \Sigma, \ x \in \bar{\Omega}\}, \]
\[ \bar{u}(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x,t) \, dx, \]
\[ \bar{f}(t,u) = \frac{1}{|\Omega|} \int_{\Omega} f(x,t,u) \, dx, \]
\[ \sigma = d\lambda - M\epsilon^2, \]
where \( \lambda > 0 \) is the principal eigenvalue of \(-\Delta\) on \(\Omega\) with homogeneous Neumann boundary condition, \(|\Omega|\) is the measure of \(\Omega\).

Our main result is

**Theorem.** Assume (A-1)-(A-3) hold, \( u_0(x) \in \Sigma \) and \( \sigma > 0 \). Then the solution \( u(x,t) \) of (1.1)-(1.3) satisfies

\[
\|u(x,t) - \bar{u}(t)\|_{L^2(\Omega)} \leq \lambda^{-1/2}\|\nabla u_0\|_{L^2(\Omega)} e^{-\sigma t/2\epsilon^2} + \lambda^{-1/2} N(t) (\epsilon^2/\sigma)|\Omega|^{1/2}, \quad t \in \mathbb{R}^+, \quad (2.1)
\]

and \( \bar{u}(t) \) satisfies

\[
\begin{cases}
\frac{d\bar{u}(t)}{dt} = \bar{f}(t,\bar{u}) + p(t), & t \in \mathbb{R}^+, \\
\bar{u}(0) = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \, dx,
\end{cases} \quad (2.2)
\]

where

\[
|p(t)| \leq M\lambda^{-1/2}|\Omega|^{-1/2}\|\nabla u_0\|_{L^2(\Omega)} e^{-\sigma t/2\epsilon^2} + MN(t)\lambda^{-1/2}\sigma^{-1}\epsilon^2, \quad t \in \mathbb{R}^+. \quad (2.3)
\]

**Proof.** Here we will use the following facts, the proof of which can be found in Appendix A of [2]:

For \( w \in W^2_2(\Omega) \), \( \partial w/\partial n|_{\partial\Omega} = 0 \), we have

\[
||\Delta w||_{L^2(\Omega)} \geq \lambda ||\nabla w||_{L^2(\Omega)}, \quad (2.4)
\]

\[
||\nabla w||_{L^2(\Omega)} \geq \lambda ||w - \bar{w}||_{L^2(\Omega)}, \quad (2.5)
\]

Define

\[
\varphi(t) = \frac{1}{2}||\nabla u||_{L^2(\Omega)}^2.
\]

Then

\[
\varphi'(t) = \int_{\Omega} \langle \nabla u, \nabla u_t \rangle \, dx = \int_{\Omega} \left\langle \nabla u, \nabla \left( \frac{1}{\epsilon^2} D\Delta u + f \right) \right\rangle \, dx
\]
\[
= -\int_{\Omega} \left\langle \Delta u, \frac{1}{\epsilon^2} D\Delta u \right\rangle \, dx + \int_{\Omega} \langle \nabla u, df_u \cdot \nabla u + df_x \rangle \, dx
\]
\[
\leq -\frac{d\lambda}{\epsilon^2} \int_{\Omega} |\nabla u|^2 \, dx + M \int_{\Omega} |\nabla u|^2 \, dx + \delta \int_{\Omega} |\nabla u|^2 \, dx + \frac{N^2}{4\delta} |\Omega|
\]
\[
= 2 \left(M - \frac{d\lambda}{\epsilon^2} + \delta\right) \varphi(t) + \frac{N^2}{4\delta} |\Omega|
\]
\[
= 2 \left(-\frac{\sigma}{\epsilon^2} + \delta\right) \varphi(t) + \frac{N^2}{4\delta} |\Omega|.
\]
Put
\[ \psi(t) = \varphi(t) + \left[ 2 \left( -\frac{\sigma}{\varepsilon^2} + \delta \right) \right]^{-1} \frac{N^2}{4\delta} |\Omega|. \]
Then we have
\[ \psi'(t) \leq 2(-\sigma/\varepsilon^2 + \delta)\psi(t) \]
and hence
\[ \psi(t) \leq \psi(0)e^{2(-\sigma/\varepsilon^2 + \delta)t}. \]
Take \( \delta = \sigma/2\varepsilon^2 \), then \( 2(-\sigma/\varepsilon^2 + \delta) = -\sigma/\varepsilon^2 < 0 \). So
\[ \varphi(t) \leq (\varphi(0) - \frac{N^2\varepsilon^4}{2\sigma^2} |\Omega|)e^{-\sigma t/\varepsilon^2} + \frac{N^2\varepsilon^4}{2\sigma^2} |\Omega| \]
\[ \leq \varphi(0)e^{-\sigma t/\varepsilon^2} + \frac{N^2\varepsilon^4}{2\sigma^2} |\Omega|, \]
i.e.,
\[ ||\nabla u||^2_{L^2(\Omega)} \leq ||\nabla u_0||^2_{L^2(\Omega)} e^{-\sigma t/\varepsilon^2} + \frac{N^2\varepsilon^4}{\sigma^2} |\Omega|, \quad t \in \mathbb{R}^+. \]
By using (2.5) we get
\[ ||u(x, t) - \bar{u}(t)||^2_{L^2(\Omega)} \leq \lambda^{-1}||\nabla u_0||^2_{L^2(\Omega)} e^{-\sigma t/\varepsilon^2} + \frac{N^2\varepsilon^4}{\sigma^2} \lambda^{-1}|\Omega|, \quad t \in \mathbb{R}^+, \]
and hence obtain (2.1).
To prove (2.2) and (2.3), we have
\[ \frac{d\bar{u}(t)}{dt} = \frac{1}{|\Omega|} \int_{\Omega} u_t(x, t) \, dx = \frac{1}{|\Omega|} \int_{\Omega} \left( \frac{1}{\varepsilon^2} D \Delta u + f \right) \, dx \]
\[ = \frac{1}{|\Omega|} \int_{\Omega} f(x, t, u) \, dx \]
\[ = \frac{1}{|\Omega|} \int_{\Omega} f(x, t, \bar{u}) \, dx + \frac{1}{|\Omega|} \int_{\Omega} (f(x, t, u) - f(x, t, \bar{u})) \, dx \]
\[ \equiv \bar{f}(t, \bar{u}) + p(t). \]
\[ |p(t)| \leq \frac{1}{|\Omega|} \int_{\Omega} |f(x, t, u) - f(x, t, \bar{u})| \, dx \]
\[ \leq \frac{M}{|\Omega|} |\Omega|^{1/2}(\int_{\Omega} |u(x, t) - \bar{u}(t)|^2 \, dx)^{1/2} \]
\[ = M|\Omega|^{-1/2}||u(x, t) - \bar{u}(t)||_{L^2(\Omega)} \]
\[ \leq M\lambda^{-1/2}|\Omega|^{-1/2}||\nabla u_0||_{L^2(\Omega)} e^{-\sigma t/2\varepsilon^2} + MN(t)\lambda^{-1/2}\sigma^{-1}\varepsilon^2, \quad t \in \mathbb{R}^+. \]
This proves (2.2) and (2.3),
The proof of the theorem is completed.

**COROLLARY 1.** If \( \lim_{t \to \infty} N(t) = 0 \), then
\[ \lim_{t \to \infty} ||u(x, t) - \bar{u}(t)||_{L^2(\Omega)} = 0, \]
(2.6)
\[ \lim_{t \to \infty} |p(t)| = 0. \]
(2.7)

*In particular, if \( D \) is a diagonal matrix, then (2.6) can be strengthened to
\[ \lim_{t \to \infty} ||u(x, t) - \bar{u}(t)||_{L^\infty(\Omega)} = 0. \]
(2.8)*
PROOF. (2.6) and (2.7) result from (2.1) and (2.3), respectively. The proof of (2.8) can be done as the same as that in Appendix B of [2].

COROLLARY 2. If \( f = f(t, u) \) instead of \( f(x, t, u) \) in (1.1), then the asymptotic behavior of the solution \( u(x, t) \) of (1.1)-(1.3) is the same as that in the case \( f = f(u) \), i.e., \( u(x, t) \) decays to \( \bar{u}(t) \) exponentially (see Theorem 3.1 of [2]).

PROOF. This is the straightforward result of our above theorem and Theorem 3.1 of [2].

3. Remark. Our theorem improves the result of Theorem 1 of [5]. First, condition \( 2d\lambda - (2M + 1)d^2 < 0 \) in [5] is weakened to \( d\lambda - Me^2 < 0 \). Second, the "error" term estimated by us is \( O(e^2) \), not merely \( O(e) \). Third, the conditions of our theorem do not depend on \( \max\{|df_x|: u \in \Sigma, x \in \bar{\Omega}, t \in \mathbb{R}^+\} \), while our conclusions deal with \( N(t) = \max\{|df_x|: u \in \Sigma, x \in \bar{\Omega}\} \). In addition, if we consider \( |\Omega| \) as a parameter, then our theorem tells that the "error" term is \( O(|\Omega|^{1/2+3/m}) \), where \( m \) is the dimension of \( x \)-space, since \( \lambda \) is inversely proportional to the squared diameter of \( \Omega \) [3].

We note as well that assumption (A-1) can be weakened to the condition that the solution of (1.1) is bounded uniformly in \( \Omega \times \mathbb{R}^+ \).

By the way, we point out that the key inequality in the proof of Theorem 2 (and hence of Theorem 3) in [5] is not true: one cannot deduce

\[
|\xi(t)| \leq \xi(0)e^{-\gamma t} = (Ne^2/\gamma)e^{-\gamma t}
\]

from

\[
d\xi(t)/dt \leq -\gamma \xi(t), \quad \xi(0) = -Ne^2/\gamma \quad ((2.5) \text{ of [5]}).
\]

The correct inequality is the converse one

\[
|\xi(t)| \geq \xi(0)e^{-\gamma t} = (Ne^2/\gamma)e^{-\gamma t},
\]

which is useless to that proof.

4. Example. To illustrate our result, let us consider a heterogeneous reaction-diffusion system of a competitor-competitor model [4]

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \frac{1}{\varepsilon^2}d_1 \frac{\partial^2 u_1}{\partial x^2} + [a_1(x) - b_1(x)u_1 - c_1(x)u_2]u_1, \\
\frac{\partial u_2}{\partial t} &= \frac{1}{\varepsilon^2}d_2 \frac{\partial^2 u_2}{\partial x^2} + [a_2(x) - b_2(x)u_2 - c_2(x)u_1]u_2,
\end{align*}
\]

with initial condition

\[
u_i(x, 0) = u_{i0}(x), \quad i = 1, 2, \; x \in [0, 1],
\]

and boundary condition

\[
\frac{\partial u_i}{\partial x}|_{x=0, 1} = 0, \quad i = 1, 2.
\]

Here \( u_1(x, t) \) and \( u_2(x, t) \) denote the population densities of two competitors with diffusion constants \( d_1/\varepsilon^2 \) and \( d_2/\varepsilon^2 \), respectively. The term

\[
[a_i(x) - b_i(x)u_i - c_1u_k]u_i \quad (i = 1, 2, \; k \neq i)
\]
represents the new growth rate of the ith competitor \( u_i \), where \( a_i(x), b_i(x), \) and \( c_i(x) \) are its intrinsic growth rate, intra- and interspecific competition coefficients, respectively. We suppose here, being the functions of spatial position \( x \), \( a_i(x), b_i(x), c_i(x) \in C^1(0,1) \), \( i = 1, 2 \).

It is easy to check that problem (4.1)-(4.3) satisfies assumptions (A-1)-(A-3). Denote

\[
L_i = \max_{x \in [0,1]} \frac{a_i(x)}{b_i(x)}, \quad i = 1, 2,
\]

then \( \Sigma = [0, L_1] \times [0, L_2] \) is an invariant region.

For simplicity, we just consider a specific case:

\[
a_1(x) = 1 - \left( x - \frac{1}{2} \right)^2, \quad a_2(x) = \frac{3}{4} + \left( x - \frac{1}{2} \right)^2, \quad b_1 = b_2 = c_1 = c_2 = 1, \quad d_1 = d_2 = 1.
\]

This means that the only difference between \( u_1 \) and \( u_2 \) is on their intrinsic growth rate functions. They prefer growing near the center and the boundary, respectively, of the habitat.

The invariant region is \( \Sigma = [0,1] \times [0,1] \) and the principal eigenvalue of \(-\Delta\) is \( \pi^2 \). In addition, we have

\[
f(x,u) = \begin{bmatrix}
(1 - (x - \frac{1}{2})^2 - u_1 - u_2)u_1 \\
\left( \frac{3}{4} + (x - \frac{1}{2})^2 - u_2 - u_1 \right)u_2
\end{bmatrix},
\]

\[
df_u = \begin{bmatrix}
1 - (x - \frac{1}{2})^2 - 2u_1 - u_2 & -u_1 \\
-2u_2 & \frac{3}{4} + (x - \frac{1}{2})^2 - 2u_2 - u_1
\end{bmatrix},
\]

\[
df_x = \begin{bmatrix}
-2(x - \frac{1}{2})u_1 \\
2(x - \frac{1}{2})u_2
\end{bmatrix},
\]

and hence

\[
N = \max_{(u_1,u_2) \in [0,1] \times [0,1]} |df_x| = \sqrt{2}.
\]

It is well known that

\[
\text{norm of } n \times n \text{ matrix } A = (\text{greatest eigenvalue of } A^T A)^{1/2}
\]

and that the greatest eigenvalue of a nonnegative definite matrix is no more than the trace of the matrix. So, by a computation we get

\[
M = \max_{(u_1,u_2) \in [0,1] \times [0,1]} |df_u|
\]

\[
\leq \max_{(u_1,u_2) \in [0,1] \times [0,1]} \left\{ \left[ 1 - (x - \frac{1}{2})^2 - 2u_1 - u_2 \right]^2 + u_1^2 + u_2^2 + \left[ \frac{3}{4} + (x - \frac{1}{2})^2 - 2u_2 - u_1 \right]^2 \right\}^{1/2}
\]

\[
\leq \sqrt{194}/4.
\]
Our theorem tells that if \( u_0(x) \in \Sigma \) and the diffusion is strong enough, namely, \( \varepsilon^2 < 4\pi^2/\sqrt{194} \), then the limit of \( \|u(x, t) - \bar{u}(t)\|_{L^2} \) is no more than
\[
\sqrt{2\varepsilon^2 \left[ \pi^2 - \sqrt{194\varepsilon^2/4} \right]}^{-1}
\]
as \( t \) tends to infinity.

REFERENCES


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