AN OSCILLATION CRITERION FOR
SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. An oscillation criterion is given for the second order nonlinear
differential equation \( x'' + a(t)|x|^{\gamma} \text{sgn } x = 0, \gamma > 0, \)
where the coefficient \( a(t) \) is not assumed to be nonnegative for all large values of \( t \).
The result involves a condition obtained by Kamenev for the linear differential equation.

Consider the second order nonlinear differential equation
\[
(1) \quad x'' + a(t)|x|^{\gamma} \text{sgn } x = 0, \quad \gamma > 0,
\]
where \( a(t) \in C[0,\infty) \). We restrict our attention to solutions of (1) which exist on
some ray \([t_0,\infty)\), where \( t_0 \geq 0 \) may depend on the particular solution. Such a
solution is said to be oscillatory if it has arbitrarily large zeros. Equation (1) is
called oscillatory if all such solutions are oscillatory. For a general discussion on
nonlinear oscillation problems, we refer the reader to [11]. We are here concerned
with sufficient conditions on \( a(t) \) for the oscillation of (1) when \( a(t) \) is allowed to
assume negative values for arbitrarily large values of \( t \). In the linear case, the
well-known Wintner's oscillation theorem states that if \( a(t) \) satisfies
\[
(2) \quad \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^t a(s) \, ds \, dt = +\infty,
\]
then equation (1) is oscillatory for \( \gamma = 1 \), see [8]. Recently, Butler [1] proved that
Wintner's theorem remains valid for equation (1) when \( \gamma > 1 \). In the sublinear
case, i.e. \( 0 < \gamma < 1 \), condition (2) can be relaxed to
\[
(3) \quad \limsup_{T \to \infty} \frac{1}{T} \int_0^T \int_0^t a(s) \, ds \, dt = +\infty,
\]
an earlier result due to Kamenev [4]. We note that condition (3) alone is not
sufficient for oscillation even in the linear case, see Willett [9]. However, if we
assume in addition to (3) the following condition
\[
(4) \quad \liminf_{t \to \infty} \int_0^t a(s) \, ds = -\lambda > -\infty, \quad \lambda > 0,
\]
then we also have oscillation in both the linear and superlinear case. For \( \gamma > 1 \),
see [10], and the linear case was settled in Hartman [3].

Several years ago, Kamenev [5] obtained an extension of Wintner's result and
proved the following oscillation criterion for the linear equation, i.e. for some \( \alpha > 1, \)
\[
(5) \quad \limsup_{T \to \infty} \frac{1}{T^\alpha} \int_0^T (T-s)^\alpha a(s) \, ds = +\infty.
\]

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It is therefore natural to ask whether condition (5) would imply oscillation also for the more general nonlinear equation (1). The purpose of this note is to prove the following:

**THEOREM.** Suppose that \( a(t) \) satisfies conditions (4) and (5), then equation (1) is oscillatory for all \( \gamma > 0 \).

**PROOF.** Assume the contrary, then there exists a solution \( x(t) \) which may be assumed to be positive on \([t_0, \infty)\) for some \( t_0 \geq 0 \). Dividing (1) through by \( x^\gamma(t) \) and integrating from \( t_0 \) to \( t \), we obtain

\[
-x^{-\gamma}(t)x'(t) + \gamma \int_{t_0}^{t} \frac{x'(s)}{x^{\beta}(s)} ds + A(t) = c_1,
\]

where \( \beta = (\gamma + 1)/2 \), \( c_1 = x^{-\gamma}(t_0)x'(t_0) \), and \( A(t) \) denotes the definite integral \( \int_{t_0}^{t} a(s) ds \).

We consider two mutually exclusive cases when \( x'x^{-\beta}(t) \in L^2(t_0, \infty) \) and \( x'x^{-\beta}(t) \notin L^2(t_0, \infty) \). In the first case, we note from Schwarz's inequality for \( f \),

\[
|x^{1-\beta}(t) - x^{1-\beta}(t_0)| = |1 - \beta| \left| \int_{t_0}^{t} \frac{x'(s)}{x^{\beta}(s)} ds \right|
\]

\[
\leq |1 - \beta|(t - t_0)^{1/2} \left( \int_{t_0}^{t} \frac{x'(s)}{x^{\beta}(s)} \right)^2 ds \right)^{1/2}.
\]

Since \( x'x^{-\beta}(t) \in L^2(t_0, \infty) \), (7) implies that there exists constants \( M \) and \( t_1 > t_0 \) such that

\[
|x^{1-\beta}(t)| \leq Mt^{1/2}, \quad t \geq t_1.
\]

Denote \( w(t) = x'(t)x^{-\gamma}(t) \), then we can rewrite equation (1) as follows:

\[
w'(t) + \gamma x^{\gamma-1}(t)w^2(t) + a(t) = 0, \quad t \geq t_1.
\]

Recall that \( 2(\beta - 1) = \gamma - 1 \), and so using (8) in (9), we obtain

\[
w'(t) + M_0t^{-1}w^2(t) + a(t) \leq 0, \quad t \geq t_1,
\]

where \( M_0 = \gamma M^{-2} \). Now we multiply (10) through by \( (T - t)^\alpha \) and integrate by parts to find

\[
\int_{t_1}^{T} M_0t^{-1}(T - t)^\alpha w^2(t) dt + \alpha \int_{t_1}^{T} (T - t)^{\alpha-1}w(t) dt
\]

\[
+ \int_{t_1}^{T} (T - t)^\alpha a(t) dt \leq (T - t_1)^\alpha w(t_1).
\]

We can combine the first two terms in (11) by completing the squares under the integral sign and obtain

\[
\int_{t_1}^{T} \left( \frac{M_0}{t} (T - t)^{\alpha/2}w(t) + \frac{\alpha}{2} \frac{t}{M} (T - t)^{\alpha/2-1} \right)^2 dt
\]

\[
- \frac{\alpha^2}{4M_0} \int_{t_1}^{T} t(T - t)^{\alpha-2} dt + \int_{t_1}^{T} (T - t)^\alpha a(t) dt \leq (T - t_1)^\alpha w(t_1).
\]
We note that the first integral in (12) is nonnegative and observe that, for \( \alpha > 1 \),
\[
\lim_{T \to \infty} \frac{1}{T^\alpha} \int_{t_1}^{T} t(T-t)^{\alpha-2} \, dt = \frac{1}{\alpha(\alpha-1)},
\]
so the second integral in (12) tends to a finite limit. Hence condition (5) would produce the desired contradiction in (12).

Next we consider \( x'x^{-\beta}(t) \notin L^2(t_0, \infty) \). Using condition (4) in (6), we deduce that \( x'(t) < 0 \) for sufficiently large \( t \), say \( t \geq t_1 \geq t_0 \). We now estimate (6) as follows:

\[
-\frac{x'(t)}{x(t)} \geq -(c_1 + \lambda) + \gamma \int_{t_1}^{t} \frac{x'^2(s)}{x^{\gamma+1}(s)} \, ds.
\]

By assumption, we can choose \( t_2 > t_1 \) such that
\[
\gamma \int_{t_1}^{t_2} \frac{x'^2(s)}{x^{\gamma+1}(s)} \, ds = 1 + c_1 + \lambda.
\]

For \( t \geq t_2 \), we multiply (13) through by the following positive term
\[
-\gamma \frac{x'(t)}{x(t)} \left\{ -(c_1 + \lambda) + \gamma \int_{t_1}^{t} \frac{x'^2(s)}{x^{\gamma+1}(s)} \, ds \right\}^{-1}
\]
and integrate from \( t_2 \) to \( t \) to obtain
\[
\log \left( -(c_1 + \lambda) + \gamma \int_{t_1}^{t} \frac{x'^2(s)}{x^{\gamma+1}(s)} \, ds \right) \geq \gamma \int_{t_2}^{t} \frac{-x'(s)}{x(s)} \, ds = \gamma \log \frac{x(t_2)}{x(t)}
\]
which together with (13) yields
\[
-x'(t)/x^{\gamma}(t) \geq x^{\gamma}(t_2)/x^{\gamma}(t),
\]
from which it follows that \( x'(t) \leq -x^{\gamma}(t_2) < 0 \), contradicting the assumption that \( x(t) > 0 \). This completes the proof.

REMARK 1. Condition (4) has also been used by Onose [7] in connection with other oscillation theorems for the sublinear equation.

REMARK 2. The same technique can be used to extend oscillation criterion for an equation with damped term, see recent results of Yan [12] and Yeh [14].

REMARK 3. Kamenev’s condition (5) is useful only when \( a(t) \) becomes negative for largest values of \( t \). If \( a(t) \) is eventually nonnegative, then (5) is equivalent to the divergence of \( A(t) \). Such results as [2, 13] are therefore not very meaningful.

REMARK 4. More recently, extensive generalizations of oscillation results for the linear equation to the nonlinear equation (1) were also given in [6].

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