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ON ISOMORPHIC CLASSICAL DIFFEOMORPHISM GROUPS. I

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ABSTRACT. Let (M_i, α_i), $i = 1, 2$, be two smooth manifolds equipped with symplectic, contact or volume forms α_i. We show that if a group isomorphism between the automorphism groups of α_i is induced by a bijective map between M_i, then this map must be a C^∞ diffeomorphism which exchanges the structures α_i. This generalizes a theorem of Takens.

1. Introduction and statement of the main results. This paper is the first of a series devoted to showing that some classical geometric structures are determined by their automorphism groups: a leitmotiv of the Erlanger Program [**6**].

Let (M_i, α_i), $i = 1, 2$, be two smooth manifolds equipped with volume forms, symplectic forms or contact forms α_i. Denote by $G_{\alpha_i}(M_i)$ the group of C^∞ diffeomorphisms of M_i, which preserve α_i in case α_i are symplectic or volume forms, or which preserve α_i up to a function in case α_i are contact forms.

Our main result is

THEOREM 1. *Let (M_i, α_i), $i = 1, 2$, be two manifolds equipped with volume forms, symplectic forms or contact forms α_i. Let $w: M_1 \to M_2$ be a bijective map such that for any map $f: M_1 \to M_1$, $wfw^{-1} \in G_{\alpha_2}(M_2)$ if and only if $f \in G_{\alpha_1}(M_1)$. Then w is a C^∞ diffeomorphism and $w^*\alpha_2 = \lambda\alpha_1$ for some function λ, which is a constant if α_i are volume or symplectic forms.*

This generalizes the following theorem of Takens [**11**].

THEOREM (TAKENS). *Let $w: M_1 \to M_2$ be a bijection between two smooth manifolds M_1 and M_2 such that $h: M_1 \to M_1$ is a C^∞ diffeomorphism iff whw^{-1} is a C^∞ diffeomorphism. Then w is a C^∞ diffeomorphism.*

We give a new and short proof of Takens' theorem. However our proof relies on a deep theorem of Montgomery-Zippin [**8**]. Then we show how the proof generalizes to the classical diffeomorphism groups mentioned in Theorem 1 to assert that w must be a C^∞ diffeomorphism. To show that this diffeomorphism exchanges the structures, we appeal to a theorem of Omori [**9**] generalizing the classical theorem of Pursell-Shanks [**10**], on Lie algebras of vector fields.

As a consequence of our main result and of a theorem of Wechsler [**13**], we obtain

THEOREM 2. *Let $G_{\alpha_i}(M_i)$, $i = 1, 2$, be as in Theorem 1. Suppose a group isomorphism $\Phi: G_{\alpha_1}(M_1) \to G_{\alpha_2}(M_2)$ is also a homeomorphism when $G_{\alpha_i}(M_i)$ are endowed with the point-open topology. Then there exists a C^∞ diffeomorphism*

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$w\colon M_1 \to M_2$ such that $\Phi(f) = wfw^{-1}$, $\forall f \in G_{\alpha_1}(M_1)$ and $w^*\alpha_2 = \lambda\alpha_1$ for some function λ which is a constant if α_i are volume or symplectic forms.

COROLLARY. *If an automorphism of the group of contact diffeomorphisms $G_\alpha(M)$ of a contact manifold (M,α) is also a homeomorphism for the point-open topology, then it is an inner automorphism.*

The goal of the second part of this paper [**2**] is to show that in the case α_i are volume forms and symplectic forms (under certain conditions), the hypothesis in our Theorem 2 that Φ is a homeomorphism for the point-open topology can be dropped. This uses an approach due to Filipkiewicz [**5**] and deep theorems on the structure of volume preserving and symplectic diffeomorphism groups [**12**, **1**]. We have not been so far in the contact case, the reason being that the structure of the group of contact diffeomorphisms is not well understood.

2. A new proof of Takens' theorem. One shows first that $w\colon M_1 \to M_2$ must be homeomorphism. This is done exactly like in Takens [**11**] or Filipkiewicz [**5**]. For completeness we reproduce here the argument.

Let $\mathrm{Diff}^\infty(M)$ be the group of all C^∞ diffeomorphisms of a smooth manifold M. Let $\mathcal{A}$ denote the class of fixed subsets of elements of $\mathrm{Diff}^\infty(M)$, i.e. $\mathcal{A} = \{\mathrm{Fix}(f) | f \in \mathrm{Diff}^\infty(M)\}$ and $\mathrm{Fix}(f) = \{x \in M | f(x) = x\}$. Let $\mathcal{B}$ be the class of complements of elements of $\mathcal{A}$, i.e. $\mathcal{B} = \{B = M - A,\ A \in \mathcal{A}\}$. This is a class of open subsets of M. If $B \in \mathcal{B}$, then B is the interior of the support of some diffeomorphism. For any point x belonging to an open set U of M, it is easy to construct a diffeomorphism whose support contains x and is contained in U, i.e. there exists $B \in \mathcal{B}$ with $x \in B$ and $B \subset U$. This means that $\mathcal{B}$ is a basis for the topology of M. If $h \in \mathrm{Diff}^\infty(M_2)$ then $\mathrm{Fix}(whw^{-1}) = w(\mathrm{Fix}(h))$ and if $g \in \mathrm{Diff}^\infty(M_2)$ $\mathrm{Fix}(w^{-1}gw) = w^{-1}(\mathrm{Fix}(g))$. Hence w and w^{-1} take basic open sets into basic open sets: they are hence both continuous, i.e. w is a homeomorphism.

We now want to show that w and w^{-1} are C^∞ maps. Let $C^\infty(M)$ denote the set of C^∞ real valued functions on a smooth manifold M. To prove that the continuous maps w, w^{-1} are C^∞, i.e. that w is a C^∞ diffeomorphism it is enough to show that $f \circ w \in C^\infty(M_1), \forall f \in C^\infty(M_2)$ and $g \circ w^{-1} \in C^\infty(M_2)$, $\forall g \in C^\infty(M_1)$ [**4**]. But the situation is symmetrical, so it is enough to show that $f \circ w \in C^\infty(M_1)$, $\forall f \in C^\infty(M_2)$.

The goal of the remainder of the proof is to provide explicit formulas for the partial derivatives of $f \circ w$.

Let $x \in M_1$ and U be an open neighborhood of x which is the domain of a local chart $\varphi\colon U \to \mathbf{R}^n$ (dimension of M_1 is n). The tangent bundle over U, $TM|_U$, is trivial and let $\partial/\partial x_i$, $i = 1, \dots, n$, be the natural basis of any T_yM_1, $\forall y \in U$. Let $X_i, i = 1, 2, \dots n$, be C^∞ vector fields on M_1, with compact supports and which coincide with $\partial/\partial x_i$ on a neighborhood of x. To get X_i, simply multiply $\partial/\partial x_i$ with a C^∞ function with compact support in U and which is equal to 1 near x.

Let h_t^i be the 1-parameter group of diffeomorphisms generated by X_i. Then $\bar{h}_t^i = \varphi h_t^i \varphi^{-1}\colon \varphi(U) \subseteq \mathbf{R}^n \to \varphi(U) \subseteq \mathbf{R}^n$ is given near x by

$$\bar{h}_t^i(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{i-1}, x_i + t, x_{i+1}, \dots, x_n).$$

Denote by $\overline{f \circ w}$ the local expression of $f \circ w$ in the chart (φ, U), i.e. $\overline{f \circ w} = f \circ w \circ \varphi^{-1}\colon \varphi(U) \to \mathbf{R}$. For $a \in U$ set $\varphi(a) = (a_1, a_2, \dots, a_n) = \bar{a}$. Then, if $a \in U$

is near x:

$$\begin{aligned}\partial/\partial x_i(\overline{f \circ w})(\bar{a}) &= \lim_{t\to 0} \frac{(f \circ w \circ \varphi^{-1})(\bar{h}_t^i(\bar{a})) - (f \circ w \circ \varphi^{-1})(\bar{a})}{t} \\ &= \lim_{t\to 0} \frac{(f \circ w)(h_t^i(a)) - (f \circ w)(a)}{t} \\ &= \frac{d}{dt}(f \circ w)(h_t^i(a))|_{t=0}.\end{aligned}$$

Let now X be any C^∞ vector field on M_1 with compact support and let h_t be the 1-parameter group of diffeomorphisms generated by X. For each t, $H_t = wh_tw^{-1}$ is a C^∞ diffeomorphism by hypothesis, and, the evaluation map:

$$\mathcal{H}: \mathbf{R} \times M_2 \to M_2: (t, x) \to H_t(x) = wh_tw^{-1}(x)$$

is continuous. Moreover $H_0 =$ identity and $H_{t+s} = H_t \circ H_s$. Therefore $\mathcal{H}: \mathbf{R} \times M_2 \to M_2(t, x) \to H_t(x)$ is a continuous action of $\mathbf{R}$ on M_2 by C^∞ diffeomorphisms. By Theorem 3, §5.2 of Montgomery-Zippin [**8**, p. 212], since $\mathbf{R}$ is a Lie group, this action is C^∞, i.e. $\mathcal{H}$ is smooth in both variables t and x.

Therefore, the 1-parameter group of diffeomorphisms H_t has an infinitesimal generator: i.e. a C^∞ vector field X_w with compact support such that $dH_t(x)/dt = X_w(H_t(x))$. Given $f \in C^\infty(M_2)$, its directional derivative $X_w \cdot f$ is a C^∞ function. (Interpret X_w as a derivation of the algebra $C^\infty(M_2)$.)

For any $x \in M_1$, we have

$$\begin{aligned}(X_w \cdot f)(w(x)) &= \frac{d}{dt} f(H_t(w(x)))|_{t=0} \\ &= \frac{d}{dt}(f \circ w)(h_t(x))|_{t=0}.\end{aligned}$$

Therefore if X is any of the vector fields X_i above, we get, $\forall a \in U$:

$$\begin{aligned}\frac{\partial}{\partial x_i}(\overline{(f \circ w)})(\bar{a}) &= \frac{d}{dt}(f \circ w)(h_t^i(a))|_{t=0} \\ &= ((X_i)_w \cdot f)(w(a)).\end{aligned}$$

This formula shows that $f \circ w$ is a C^1 map, and that for any vector fields X on M_1 with compact support, we have

$$(X_w \cdot f) \circ w = X \cdot (f \circ w).$$

To compute higher partial derivatives, just iterate this formula using vector fields X_i. For instance

$$\begin{aligned}\{(X_j)_w \cdot [(X_i)_w \cdot f]\} \circ w &= X_j \cdot [((X_i)_w \circ f) \circ w] \\ &= X_j \cdot (X_i \cdot (f \circ w)).\end{aligned}$$

Therefore we have proved that $f \circ w \in C^\infty(M_1)$. This completes the proof of Takens' theorem. □

REMARK. Starting with a C^∞ vector field with compact support X on M_1 and if h_t is its 1-parameter group of diffeomorphisms, we have seen that thanks to Montgomery-Zippin, $H_t = wh_tw^{-1}$ is a 1-parameter group of diffeomorphisms, which is generated by X_w. Since w is a C^∞ diffeomorphism, we must have $X_w = w_*X$ i.e. $(w_*X)(y) = (T_{w^{-1}(y)}w)(X(w^{-1}(y))), \forall y \in M_2$. Thus w induces a Lie algebra isomorphism between the Lie algebras of compactly supported vector fields on M_1 and M_2 by $X \to X_w$.

3. Generalizations: Proof of Theorem 1. For any point x belonging to an open set U of M_i it is easy to construct an $h \in G_{\alpha_i}(M_i)$ such that $x \in \mathrm{Int}(\mathrm{supp}(h))$ and $\mathrm{supp}(h) \subset U$. Therefore, the same argument as in the proof of Takens' theorem shows that w must be a homeomorphism.

Let $\mathcal{L}_{\alpha_i}(M_i)$ be the Lie algebra of vector fields with compact supports on M_i, generating 1-parameter groups of diffeomorphisms h_t belonging to $G_{\alpha_i}(M_i)$. For each $X \in \mathcal{L}_{\alpha_1}(M_1)$ and $f \in C^\infty(M_2)$, we have

$$(*) \qquad \frac{d}{dt}(f \circ w)(h_t(x))|_{t=0} = (X_w \cdot f)(w(x))$$

where X_w is defined as in the proof of Takens' theorem and h_t is the 1-parameter group of diffeomorphisms generated by X.

As before, we want to use $(*)$ to compute partial derivatives of $f \circ w$.

We consider first the volume preserving and symplectic cases. Let $x \in M_1$ and U a contractible open neighborhood of x which is the domain of a local canonical chart $\varphi: U \to \mathbf{R}^n$ ($n = \dim M_1$): in this chart, $\alpha_1|_U = \varphi^*(\underline{\alpha})$ where

$$\underline{\alpha} = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2m-1} \wedge dx_{2m} \qquad (n = 2m)$$

in the symplectic case and

$$\underline{\alpha} = dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n$$

in the volume case. The existence of these canonical charts are guaranteed by Darboux's theorem.

The vector fields $(\partial/\partial x_k)$ defined on U can be extended into elements of $\mathcal{L}_{\alpha_1}(M_1)$ [**9**]. Indeed, clearly $L_{(\partial/\partial x_k)}\alpha_1 = 0$, when $L_y\alpha$ is the Lie derivative of α in the direction of a vector field y. Hence $di(\partial/\partial x_k)\alpha_1 = 0$, here $\mathrm{i}(X)\alpha$ is the interior product of α and X. By Poincaré's lemma, there exists a function (symplectic case) or an $(n-2)$-form (volume case) $\mathcal{B}_k$ with: $\mathrm{i}(\partial/\partial x_k)\alpha_1 = d\mathcal{B}_k$. Let λ be a smooth function with compact support in U and $\lambda = 1$ near x. The vector fields X_k extending $\partial/\partial x_k$ may then be defined by

$$\mathrm{i}(X_k)\alpha_1 = d(\lambda \mathcal{B}_k).$$

Therefore in these cases we can use $(*)$ to get all the partial derivatives of $f \circ w$.

In the contact case, the canonical chart around $x, \varphi: U \to \mathbf{R}^n$ ($n = 2m+1$) is such that $\alpha_1|_U$ is a multiple of $\varphi^*\underline{\alpha}$ with

$$\underline{\alpha} = dz - (y_1 dx_1 + y_2 dx_2 + \cdots + y_m dx_m).$$

On U consider the vector fields given by

$$Z = \partial/\partial z, \quad X_k = \partial/\partial x_k, \quad Y_k = \partial/\partial y_k + x_k \partial/\partial z.$$

Clearly, if ξ is one of the vector fields above, $L_\xi \underline{\alpha} = 0$. We now want to show that these vector fields defined on U can be extended into elements of $\mathcal{L}_{\alpha_1}(M_1)$ [**9**].

It is well known that a contact vector field ξ on a contact manifold (M, α) is completely determined by the function $\mathrm{i}(\xi)\alpha$ [**7**]. Therefore if λ is a C^∞ function which is equal to 1 near x and has compact support in U, the function $\lambda(\mathrm{i}(\xi)\alpha)$ (where ξ is one of the vector fields above) determines contact vector fields $\overline{Z}, \overline{X}_k, \overline{Y}_k$ which have compact supports and coincide with Z, X_k, Y_k near x. As in the proof of Takens' theorem, we can compute the partial derivatives of $\overline{f \circ w}$, the local expansion of $f \circ w$ within the canonical charts as follows:

$$\begin{aligned} \partial/\partial z(\overline{f \circ w}) &= (\overline{Z}_w \cdot f)w(x), \\ \partial/\partial x_k(\overline{f \circ w}) &= ((\overline{X}_k)_w \cdot f)w(x), \\ \partial/\partial y_k(\overline{f \circ w}) &= ((\overline{Y}_k)_w \cdot f)w(x) - x_k((\overline{Z}_w) \cdot f)w(x). \end{aligned}$$

To compute higher derivatives, apply again these formulas with f replaced by $\overline{Z}_w \cdot f, (\overline{X}_k)_w \cdot f, (\overline{Y}_k)_w \cdot f$, etc. We thus have proved that $f \circ w$ is C^∞ and hence w is a C^∞ diffeomorphism.

The last task is now to show that w exchanges the structures.

As we have already observed, $\forall X \in \mathcal{L}_{\alpha_1}(M_1)$ we get $X_w \in \mathcal{L}_{\alpha_2}(M_2)$ such that $X_w = w_* X$. Therefore w induces a Lie algebra isomorphism between the Lie algebras $\mathcal{L}_{\alpha_i}(M_i)$.

By a theorem of Omori [**9**], generalizing a classical result of Pursell-Shanks [**10**], there exists a C^∞ diffeormorphism $\rho: M_1 \to M_2$ inducing the isomorphism $X \to X_w = w_* X$ and such that $\rho^*\alpha_2 = \lambda\alpha_1$ for some function λ which is a constant if α_i are symplectic or volume forms. But the condition $\rho_*\xi = w_*\xi$, $\forall \xi \in \mathcal{L}_{\alpha_1}(M_1)$ implies that $\rho = w$. Indeed if $\varphi = \rho^{-1}w$ and h_t is the 1-parameter group of diffeomorphisms generated by ξ, then $\varphi h_t \varphi^{-1} = h_t$. But this implies that φ=identity, i.e. $\rho = w$. Indeed if $\varphi \neq$ identity, let $x \in M_1$ with $\varphi(x) = y \neq x$. Choose $\xi \in \mathcal{L}_{\alpha_1}(M_1)$ with $\xi(x) \neq 0$, with a support not containing y. If h is the time-one flow of ξ, then: $h(x) = z \neq x$ (since $\xi(x) \neq 0$) and $\varphi h \varphi^{-1}(y) = \varphi h(x) = \varphi(z)$. But $h(y) = y = \varphi(x) \neq \varphi(z)$ since $x \neq z$, i.e. $\varphi h \varphi^{-1} \neq h$. Contradiction. The proof of Theorem 1 is now complete. □

4. Applications. A group of homeomorphisms $G(X)$ of a topological space X is said to be ω-transitive on X if for each $n \in \mathbf{N}$ and each pair on n-tuples of distinct points $(x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)$, there exists a homeomorphism $h \in G(X)$ such that $h(x_i) = y_i$.

In [**13**], Wechsler has proved a theorem which contains the following result:

THEOREM (WECHSLER). *Let $\phi: G(M_1) \to G(M_2)$ be a group isomorphism between two groups of diffeomorphisms of two smooth connected manifolds M_i, of dimension $n \geq 2$. Suppose that $G(M_i)$ are ω-transitive on M_i and that ϕ is a homeomorphism when $G(M_i)$ are endowed with the point-open topology. Then, there exists a homeomorphism $w: M_1 \to M_2$ such that $\phi(f) = w \circ f \circ w^{-1}$, $\forall f \in G(M_1)$.*

PROOF OF THEOREM 2. By Boothby's theorem [**3**], the groups considered in Theorem 2 are ω-transitive. Therefore by Wechsler's theorem, there exists a homeomorphism $w: M_1 \to M_2$ such that $\varphi(f) = wfw^{-1}, \forall f \in G_{\alpha_1}(M_1)$. By our Theorem 1, w is a C^∞ diffeomorphism and exchanges the structures α_i. □

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