A LOCAL STRONG \( UV^\infty \)-PROPERTY
OF THE HOMEOMORPHISM GROUPS
OF \( R^\infty \) - (\( Q^\infty \))- MANIFOLDS

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Abstract. We will show in this note that if \( M \) is an \( R^\infty \) - (or \( Q^\infty \))- manifold having the homotopy type of a finite complex, then the homeomorphism group of \( M \), endowed with the compact-open topology, has the local strong \( UV^\infty \)-property with respect to the classes of pseudo CW complexes \( \mathcal{C} \mathcal{W}(\mathcal{C}) \) and \( \mathcal{C} \mathcal{W}(\mathcal{M}) \).

Let \( E^\infty = \limdir E^n \) where \( E^n \) is either the \( n \)-Euclidean space or the \( n \)-fold product of the Hilbert cube \( Q \). By an \( E^\infty \)-manifold, we mean a paracompact space which is locally homeomorphic to \( E^\infty \). Given a space \( X \), let \( \text{Homeo}(X) \) denote the group of homeomorphisms of \( X \) endowed with the compact-open topology. Let \( A \) be a subset of \( S \times X \) (or \( X \times S \)), a map \( f: A \to S \times Y \) (or \( f: A \to Y \times S \)) is said to be \( S \)-fiber preserving (\( S \)-f.p.) if \( \pi_S f = \pi_S \) where \( \pi_S \) denotes the projection. Given an \( S \)-f.p. map \( f: S \times X \to S \times Y \), for each \( s \in S \), let \( f_s: X \to Y \) denote the map defined by \( f(x, x) = (s, f_s(x)) \) for each \( x \in X \). By an \( S \)-f.p. isotopy (an \( S \)-f.p. invertible isotopy, resp.) from \( S \times X \) onto \( S \times Y \), we mean an \((I \times S)\)-f.p. map \( f: I \times S \times X \to I \times S \times X \) such that \( f_{t,s} \) is a homeomorphism for each \((t, s) \in I \times S \). \( f \) is an \((I \times S)\)-f.p. homeomorphism, resp.). Given an open cover \( \mathcal{U} \) of \( Y \), a homotopy \( H: I \times X \to Y \) is said to be an \( \mathcal{U} \)-homotopy if for each \( x \in X \) there is an \( U \in \mathcal{U} \) such that \( H(I \times x) \subseteq U \). Two maps \( f, g: X \to Y \) are said to be \( \mathcal{U} \)-close if for each \( x \in X \), there is an \( U \in \mathcal{U} \) such that \( \{ f(x), g(x) \} \subseteq U \). Following [L], let \( \mathcal{C} \mathcal{W}(\mathcal{C}) \) and \( \mathcal{C} \mathcal{W}(\mathcal{M}) \) denote the classes of pseudo CW complexes generated by the class \( \mathcal{C} \) of Hausdorff compact spaces and the class \( \mathcal{M} \) of metric spaces, respectively. If \( A \) is a subset of \( X \), let \( i_A \) denote the inclusion \( A \hookrightarrow X \), and \( \text{id}_X \) the identity of \( X \).

The homeomorphism group \( \text{Homeo}(X) \) of a space \( X \) is said to have the \textit{local strong} \( UV^\infty \)-\textit{property with respect to a class} \( \mathcal{C} \) provided that given an open neighborhood \( U \) of an \( h \in \text{Homeo}(X) \), there is an open neighborhood \( V \) of \( h \) in \( U \) such that if \( \Lambda \in \mathcal{C} \), \( \Gamma \) is a closed \( G_\delta \)-set in \( \Lambda \), and \( g: \Lambda \to V \) is a continuous map with \( g(\Gamma) = \{ h \} \), then \( g \) is homotopic rel \( \Gamma \) in \( V \) to the constant map \( H: V \to \{ h \} \).

If \( M \) is an \( R^\infty \)-manifold, Hale provided in [H] that \( \text{Homeo}(M) \) is Lindelöf, homogeneous, and paracompact, but not compactly generated. In this note, we
intend to study the local contractibility of the groups of homeomorphisms of $E^\infty$-manifolds.

**Theorem 1.** If $M$ is an $E^\infty$-manifold having the homotopy type of a finite complex, then $\text{Homeo}(M)$ has the local strong UV$^\infty$-property with respect to $\mathcal{CW}(\mathcal{C}) \cup \mathcal{CW}(\mathcal{M})$.

Note that if $M = \tilde{D} \times E^\infty$ where $\tilde{D}$ is the open disk with infinitely many handles attached (refer to [R, p. 275]), then $\text{Homeo}(M)$ is not locally path-connected.

Let us introduce some notations that will be used in the following proofs. All spaces are Hausdorff. $M$ is a given $E^\infty$-manifold having the homotopy type of a compact subpolyhedron $A$ of an $R^m$. Let $N$ be a regular neighborhood of $A$ in $R^m$ [Hd] and identify $M$ with $N \times E^\infty$ [He, Theorem C] and $N$ with $N \times 0$ in $N \times E^\infty$. Referring to [H-T], write $M = \text{dlirm}_j M_j$ where $M_j = N \times B_{j+1}$ and $B_j$ denotes the $s$-fold product of $[-s,s]$ ($Q$, resp.) if $E = R$ (if $E = Q = \prod_{i=0}^{\infty}[0,1]$, resp.). Given a positive integer $s$ and a sequence $\{\epsilon_s, \epsilon_{s+1}, \ldots\}$ of positive numbers, we write $E^\infty(\epsilon_s, \epsilon_{s+1}, \ldots) = \{x \in E^\infty \mid x_1 = \cdots = x_{s-1} = 0 \text{ and } d(x_j, 0) < \epsilon_j \text{ for } j = s, s + 1, \ldots\}$.

Given a compact subset $K$ of $M$ and an open set $V$ of $M$, following [D] let $[K, V]$ denote the set $\{f \in \text{Homeo}(M) \mid f(K) \subset V\}$; this is a member of a subbasis for the compact-open topology on $\text{Homeo}(M)$. We will work with the category $\mathcal{G}$ of compactly generated spaces [G]. For $X, Y \in \mathcal{G}$, by $X \times Y$ we mean the $\mathcal{G}$-product of $X$ and $Y$. Recall that $\mathcal{CW}(\mathcal{C}) \cup \mathcal{CW}(\mathcal{M})$ is a subclass of $\mathcal{G}$ [L]; moreover, that if $\Lambda \in \mathcal{CW}(\mathcal{C}) \cup \mathcal{CW}(\mathcal{M})$, then so does $I \times \Lambda$.

**Lemma 2.** Let $\Gamma$ be a closed subset of $\Lambda \in \mathcal{CW}(\mathcal{C}) \cup \mathcal{CW}(\mathcal{M})$ and $q$ a nonnegative integer. Let $f, g: \Lambda \times M \to \Lambda \times M$ be $\Lambda$-f.p. maps such that $f|\Gamma \times M = g|\Gamma \times M$ and such that there is a $\Lambda$-f.p. homotopy $\phi$ from $f|\Lambda \times M_q$ to $g|\Lambda \times M_q$ (rel $\Gamma \times M_q$). Then, $\phi$ can be extended to a $\Lambda$-f.p. homotopy $\psi$ from $f$ to $g$ (rel $\Gamma \times M$). Moreover, if $g$ is a $\Lambda$-f.p. homotopy equivalence, then $\psi: I \times \Lambda \times M \to I \times \Lambda \times M$, defined by $\psi(t, \lambda, z) = (t, \psi(t, \lambda, z))$, is an $(I \times \Lambda)$-f.p. homotopy equivalence.

**Proof.** Let $A = (I \times \Lambda \times M_q) \cup (I \times \Gamma \times M) \cup \{(0,1) \times \Lambda \times M\}$ and $\overline{\phi}: A \to \Lambda \times M$ be the extension of $\phi$ defined by

$$
\overline{\phi}(t, \lambda, x) = \begin{cases}
\phi(t, \lambda, x) & \text{if } x \in M_q, \\
f(\lambda, x) = g(\lambda, x) & \text{if } \lambda \in \Gamma, \\
f(\lambda, x) & \text{if } t = 0, \\
g(\lambda, x) & \text{if } t = 1.
\end{cases}
$$

Since $M_q$ is the strong deformation retract of $M$, we have a homotopy $\theta: M \times I \to M$ such that $\theta_0 = \text{id}$, $\theta_1(M) = M_q$ and $\theta_t|M_q = \text{id}$ for each $t \in I$. Then, $\overline{\theta} = \text{id} \times \theta: I \times \Lambda \times M \times I \to I \times \Lambda \times M$ is an $(I \times \Lambda)$-f.p. homotopy such that $\overline{\theta}_0 = \text{id}$, $\overline{\theta}(A \times I) \subset A$, and $\overline{\theta}_1: I \times \Lambda \times M \to I \times \Lambda \times M_q$ is a $(I \times \Lambda)$-f.p. retraction. Observe that $\phi = \overline{\phi}\overline{\theta}_0|A$ is $\Lambda$-f.p. homotopic to $\overline{\phi}\overline{\theta}_1|A = \phi\overline{\theta}_1|A$ and $\phi\overline{\theta}_1|A$ has the
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By [L, Lemma 1.1], \( \phi \) has also a \( \Lambda \)-f.p. extension \( \psi : I \times \Lambda \times M \rightarrow \Lambda \times M \). This \( \psi \) is a wanted extension of \( \phi \).

Finally, it is easy to prove that \( \hat{\psi} \) is \((I \times \Lambda)\)-f.p. homotopic to \( \text{id}_I \times g : I \times \Lambda \times M \rightarrow I \times \Lambda \times M \). Therefore, \( \hat{\psi} \) is an \((I \times \Lambda)\)-f.p. homotopy equivalence if \( g \) is a \( \Lambda \)-f.p. homotopy equivalence.

**Proof of Theorem 1.** Since it is straightforward to verify that the translation map is a homeomorphism of \( \text{Homeo}(M) \) (or refer to [H]), we only need to show that \( \text{Homeo}(M) \) has the local \( UV^\infty \)-property at \( \text{id}_M \). Let \( U \) be a given neighborhood of \( \text{id}_M \) in \( \text{Homeo}(M) \) of the form \( \bigcap \{ [K_i, \Omega_i] | i = 1, \ldots, p \} \) where \( K_i \) is a compact subset of the open subset \( \Omega_i \) in \( M \) for each \( i \).

**Definition of \( V \).** Without loss of generality, we can assume that \( K_1 \cup \cdots \cup K_p \subset M_0 \) [Ha, Lemma 2.4]. Then, since \( M_0 \) is locally convex, there is a finite cover \( \mathcal{A} \) of \( M_0 \) consisting of convex open sets in \( M_1 \) such that

(a) \( \text{For } E = R, \text{ if } A \in \mathcal{A} \text{ with } A \cap M_0 \neq \emptyset, \text{ then } A \subset N \times \text{Int } B_2. \)

(b) \( \text{If } A \in \mathcal{A} \text{ with } A \cap K_i \neq \emptyset, \text{ then its closure } \overline{A} \subset \Omega_i. \)

Note that any two \( \mathcal{A} \)-close maps into \( M_1 \) are canonically \( \mathcal{A} \)-homotopic. Let \( \delta > 0 \) be a Lebesgue number for the open cover \( \{ A \cap M_0 | A \in \mathcal{A} \} \) of \( M_0 \), and let \( \mathcal{F} \) be a finite closed cover of \( M_0 \) each of whose members has diameter less than \( \delta \). Since the union \( \bigcup \{ \overline{A} | A \in \mathcal{A} \text{ and } \overline{A} \subset \Omega_i \} \) is compact for each \( i = 1, \ldots, p \), there is a sequence of positive numbers \( \varepsilon_2, \varepsilon_3, \ldots \) such that for each \( i = 1, \ldots, p \) if \( \overline{A} \subset \Omega_i \), then \( A \times E_{(\varepsilon_2, \varepsilon_3, \ldots)} \subset \Omega_i. \) Now, for each \( C \in \mathcal{F}, \) since \( \text{diam}(C) < \delta \), there is an \( A \in \mathcal{A} \) such that \( C \subset A \). Define

\[ V = \bigcap \{ (A \times E_{(\varepsilon_2, \varepsilon_3, \ldots)} | C \in \mathcal{F}) \}. \]

(For each \( C \in \mathcal{F}, \) \( A \times E_{(\varepsilon_2, \varepsilon_3, \ldots)} \) is an open set in \( M \) by use of (a).) Observe that \( V \) is an open neighborhood of \( \text{id}_M \) in \( U \). Let \( f \in V \). Fix an \( i \) \((i = 1, \ldots, p)\) and let \( x \in K_i. \) Then, there is a \( C \in \mathcal{F} \) such that \( x \in C. \) So, \( A \cap K_i \neq \emptyset; \) consequently, it follows from (b) that \( C \subset A \subset C \subset \Omega_i. \) Therefore, \( f(K_i) \subset \Omega_i \) for each \( i; \) so, \( f \) \( \in U. \) Moreover, that if \( f \in \text{Homeo}(M), \) \( g \in V \) and \( f \cap M_0 = g \cap M_0, \) then \( f \in V. \)

**The local strong \( UV^\infty \)-property.** Let \( \Lambda \in \mathcal{CW}(\mathcal{E}) \cup \mathcal{CW}(\mathcal{M}) \) and \( \Gamma \) a closed \( G_\delta \)-set in \( \Lambda. \) Let \( g : \Lambda \rightarrow V \) be a continuous map with \( g(\Gamma) = \{ \text{id}_M \}. \) We will prove that \( g \) is homotopic rel \( \Gamma \) in \( V \) to the constant map \( \text{Id} \) with \( \text{Id}(\Lambda) = \{ \text{id}_M \}. \)

Let \( f : \Lambda \times M \rightarrow \Lambda \times M \) be the \( \Lambda \)-f.p. map associated to \( g, \) defined by \( f_\lambda(z) = g(\lambda)(z) \) for all \((\lambda, z) \in \Lambda \times M, \) where \( \Lambda \times M \) is the \( \mathcal{G}\mathcal{E} \)-product space as used in [L]. Then, \( f \) is continuous by [G, Theorem 8.17]. By [D, Theorem XII.3.1(1)] for the \( \mathcal{G}\mathcal{E} \)-product, we only need to prove that there is a \( \Lambda \)-f.p. isotopy (rel \( \Gamma \times M \)) \( F : I \times \Lambda \times M \rightarrow I \times \Lambda \times M \) with \( F_0 = f, \) \( F_1 = \text{id}_{\Lambda \times M} \) and \( F_{(t, \lambda)} \in V \) for each \((t, \lambda) \in I \times \Lambda. \)

We will define by induction a sequence \( \{ F^n | n = 1, 2, \ldots \} \) of \( \Lambda \)-f.p. invertible isotopies of \( \Lambda \times M \) such that

(0) \( F^1_1 = \text{id}_M \) and \( F^1_{(t, \lambda)} \in V \) for each \((t, \lambda) \in I \times \Lambda, \)

(1) \( F^n_1 = F^{n-1}_0 (n > 1), \)

(2) \( F^n_0 | \Lambda \times M_{n-1} = f | \Lambda \times M_{n-1} (n > 0), \)

(3) \( F^n_0 | \Lambda \times M_{n-2} = f | \Lambda \times M_{n-2} \) for each \( t \in I, n > 1, \) and

(4) \( F^n_0 | \Gamma \times M = \text{id}_{\Gamma \times M}. \)
Next, define \( F: I \times \Lambda \times M \to I \times \Lambda \times M \) by

\[
F(t, \lambda, z) = \begin{cases} 
(0, f(\lambda, z)) & \text{if } t = 0, \\
(t, F^n(2^n t - 1, \lambda, z)) & \text{if } 1/2^n \leq t \leq 1/2^{n-1}.
\end{cases}
\]

Then, \( F \) will be an \((I \times \Lambda)\)-f.p. isotopy. It is well defined by \((1)_n\) and continuous by \((2)_n\), \((3)_n\) and \( I \times \Lambda \times M = \text{dirlim}(I \times \Lambda \times M_n) \) [L, Lemma 0.3]. It also is clear that \( F_0 = f \) and that \( F_1 = F^1_{i\Lambda \times M} \) by \((0)\). Moreover, each \( F_{t, \lambda} \in V \) by use of \((0)\) and \((3)_n\). Therefore, the proof will be complete. (Note that \( F \) is not necessary an invertible isotopy.)

First, let \( H: I \times \Lambda \times M_0 \to \Lambda \times M \) be the straight-line homotopy from \( f \mid \Lambda \times M_0 \) to \( i_{\Lambda \times M_0} \). Observe that \( H \) is a \( \Lambda \)-f.p. homotopy rel \( \Gamma \times M_0 \) with \( H(t \times \lambda \times C) \subset A_C \times E(\epsilon_2, \epsilon_3, \ldots) \) for each \( C \in \mathcal{C} \). Let \( \hat{H}: I \times \Lambda \times M_0 \to I \times \Lambda \times M \) be the \((I \times \Lambda)\)-f.p. map defined by \( \hat{H}(t, \lambda, x) = (t, H(t, \lambda, x)) \). By [L, Lemma 1.2], we can assume that \( \hat{H} \) is an \((I \times \Lambda)\)-f.p. embedding. Moreover, following the proof of [L, Lemma 1.2], to adjust only the \( E(\epsilon_2, \epsilon_3, \ldots) \)-component of \( \hat{H} \) (rel \( \partial I \times \Lambda \times M_0 \)), we can assume that \( \pi_{\Lambda \times M} \hat{H} \) is still a \( \Lambda \)-f.p. homotopy rel \( \Gamma \times M_0 \) with

\[
\pi_{\Lambda \times M} \hat{H}(t \times \lambda \times C) \subset A_C \times E(\epsilon_2, \epsilon_3, \ldots)
\]

for each \( C \in \mathcal{C} \). By Lemma 2, \( \hat{H} \) has an \((I \times \Lambda)\)-f.p. homotopy-equivalence extension \( \hat{\psi}^1: I \times \Lambda \times M \to I \times \Lambda \times M \) such that

\[
\begin{align*}
(a)_1 \hat{\psi}^1_2 &= \text{id}_{\Lambda \times M}, \\
(b)_1 \hat{\psi}^0_0 &= f, \text{ and} \\
(c)_1 \hat{\psi}^0_1|\Gamma \times M &= i_{\Gamma \times M} \text{ for each } t \in I.
\end{align*}
\]

Therefore, by [L, Lemma 3.3], there is an \((I \times \Lambda)\)-f.p. homeomorphism \( F^1 \) of \( I \times \Lambda \times M \) such that \( F^1 \) is an extension of \( \hat{\psi}^1|\{I \times \Lambda \times M_0\} \cup \{I \times \Gamma \times M\} \cup \{1 \times \Lambda \times M\} \). Now, the properties \((2)_1, (4)_1\) follow from \((b)_1\) and \((c)_1\), respectively. To verify \((0)\), it is clear that \( F^1_0 = \text{id}_{\Lambda \times M} \) by \((a)_1\), and that \( F^1_{t, \lambda} \in V \) by \((*)\) since \( \hat{\psi}^1 \) is an extension of \( \hat{H} \).

Second, consider \( f \) and \( F^0_0 \). Since \( f \mid \Lambda \times M_0 = F^0_0 \mid \Lambda \times M_0 \), by use of Lemma 2, we can obtain a \( \Lambda \)-f.p. homotopy \( \phi: f \mid \Lambda \times M_1 = F^0_0 \mid \Lambda \times M_1 \) rel \( \{\Gamma \times M_1\} \cup \{\Lambda \times M_0\} \). By [L, Lemma 1.2], we can assume that \( \phi: I \times \Lambda \times M_1 \to I \times \Lambda \times M \), defined by \( \phi(t, \lambda, z) = (t, \phi(t, \lambda, z)) \), is an \((I \times \Lambda)\)-f.p. embedding. By Lemma 2, \( \hat{\phi} \) has an \((I \times \Lambda)\)-f.p. homotopy equivalence extension \( \hat{\psi}^2: I \times \Lambda \times M \to I \times \Lambda \times M \) such that

\[
\begin{align*}
(a)_2 \hat{\psi}^2_1 &= F^1_0, \\
(b)_2 \hat{\psi}^0_0 &= f, \text{ and} \\
(c)_2 \hat{\psi}^0_1|\Gamma \times M &= i_{\Gamma \times M}.
\end{align*}
\]

Then, by [L, Lemma 3.3], there is an \((I \times \Lambda)\)-f.p. homeomorphism \( F^2 \) of \( I \times \Lambda \times M \) such that \( F^2 \) is an extension of \( \hat{\psi}^2|\{I \times \Lambda \times M_1\} \cup \{I \times \Gamma \times M\} \cup \{1 \times \Lambda \times M\} \). The properties \((1)_2, (2)_2\) and \((4)_2\) follow from \((a)_2, (b)_2\) and \((c)_2\), respectively. The property \((3)_2\) holds true since \( \hat{\psi}^2 \) is an extension of \( \hat{\phi} \).

Finally, in a similar manner, we can define a wanted \((I \times \Lambda)\)-f.p. homeomorphism \( F^n \) of \( I \times \Lambda \times M \) when an appropriate invertible isotopy \( F^{n-1} \) has already been defined. Therefore, the proof is complete.  

The author wishes to thank the referee for his suggestion to simplify the proof of Lemma 2.

REFERENCES


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