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A LOCAL STRONG UV^∞-PROPERTY OF THE HOMEOMORPHISM GROUPS OF R^∞- (Q^∞-) MANIFOLDS

VO THANH LIEM

ABSTRACT. We will show in this note that if M is an R^∞- (or Q^∞-) manifold having the homotopy type of a finite complex, then the homeomorphism group of M, endowed with the compact-open topology, has the local strong UV^∞-property with respect to the classes of pseudo CW complexes $\mathscr{CW}(\mathscr{C})$ and $\mathscr{CW}(\mathscr{M})$.

Let $E^\infty = \operatorname{dirlim} E^n$ where E^n is either the n-Euclidean space or the n-fold product of the Hilbert cube Q. By an *E^∞-manifold*, we mean a paracompact space which is locally homeomorphic to E^∞. Given a space X, let Homeo(X) denote the group of homeomorphisms of X endowed with the compact-open topology. Let A be a subset of $S \times X$ (or $X \times S$), a map $f\colon A \to S \times Y$ (or $f\colon A \to Y \times S$) is said to be *S-fiber preserving* (S-f.p.) if $\pi_S f = \pi_S$ where π_S denotes the projection. Given an S-f.p. map $f\colon S \times X \to S \times Y$, for each $s \in S$, let $f_s\colon X \to Y$ denote the map defined by $f(x, x) = (s, f_s(x))$ for each $x \in X$. By an S-f.p. isotopy (an S-f.p. invertible isotopy, resp.) from $S \times X$ onto $S \times Y$, we mean an $(I \times S)$-f.p. map $f\colon I \times S \times X \to I \times S \times X$ such that $f_{(t,s)}$ is a homeomorphism for each $(t, s) \in I \times S$ (f is an $(I \times S)$-f.p. homeomorphism, resp.). Given an open cover $\mathscr{U}$ of Y, a homotopy $H\colon I \times X \to Y$ is said to be an *$\mathscr{U}$-homotopy* if for each $x \in X$ there is an $U \in \mathscr{U}$ such that $H(I \times x) \subset U$. Two maps $f, g\colon X \to Y$ are said to be *$\mathscr{U}$-close* if for each $x \in X$, there is an $U \in \mathscr{U}$ such that $\{f(x), g(x)\} \subset U$. Following [L], let $\mathscr{CW}(\mathscr{C})$ and $\mathscr{CW}(\mathscr{M})$ denote the classes of pseudo CW complexes generated by the class $\mathscr{C}$ of Hausdorff compact spaces and the class $\mathscr{M}$ of metric spaces, respectively. If A is a subset of X, let i_A denote the inclusion $A \hookrightarrow X$, and id_X the identity of X.

The homeomorphism group Homeo(X) of a space X is said to have the *local strong UV^∞-property with respect to a class $\mathscr{X}$* provided that given an open neighborhood U of an $h \in$ Homeo(X), there is an open neighborhood V of h in U such that if $\Lambda \in \mathscr{X}$, Γ is a closed G_δ-set in Λ, and $g\colon \Lambda \to V$ is a continuous map with $g(\Gamma) = \{h\}$, then g is homotopic rel Γ in V to the constant map $H\colon V \to \{h\}$.

If M is an R^∞-manifold, Hale provided in [H] that Homeo(M) is Lindelöf, homogeneous, and paracompact, but not compactly generated. In this note, we

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intend to study the local contractibility of the groups of homeomorphisms of E^∞-manifolds.

THEOREM 1. *If M is an E^∞-manifold having the homotopy type of a finite complex, then* Homeo(M) *has the local strong UV^∞-property with respect to $\mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$.*

Note that if $M = \tilde{D} \times E^\infty$ where $\tilde{D}$ is the open disk with infinitely many handles attached (refer to [**R**, p. 275]), then Homeo(M) is not locally path-connected.

Let us introduce some notations that will be used in the following proofs. All spaces are Hausdorff. M is a given E^∞-manifold having the homotopy type of a compact subpolyhedron A of an R^m. Let N be a regular neighborhood of A in R^m [**Hd**] and identify M with $N \times E^\infty$ [**He**, Theorem C] and N with $N \times 0$ in $N \times E^\infty$. Referring to [**H-T**], write $M = \operatorname{dirlim} M_j$ where $M_j = N \times B_{j+1}$ and B_s denotes the s-fold product of $[-s, s]$ (Q, resp.) if $E = R$ (if $E = Q = \prod_1^\infty [0,1]$, resp.). Given a positive integer s and a sequence $\{\varepsilon_s, \varepsilon_{s+1}, \dots\}$ of positive numbers, we write $E^\infty_{\{\varepsilon_s, \varepsilon_{s+1}, \dots\}} = \{x \in E^\infty \mid x_1 = \cdots = x_{s-1} = 0$ and $d(x_j, 0) < \varepsilon_j$ for $j = s, s+1, \dots\}$.

Given a compact subset K of M and an open set V of M, following [**D**] let $[K, V]$ denote the set $\{f \in \text{Homeo}(M) \mid f(K) \subset V\}$; this is a member of a subbasis for the compact-open topology on Homeo(M). We will work with the category $\mathscr{CG}$ of compactly generated spaces [**G**]. For $X, Y \in \mathscr{CG}$, by $X \times Y$ we mean the $\mathscr{CG}$-product of X and Y. Recall that $\mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$ is a subclass of $\mathscr{CG}$ [**L**]; moreover, that if $\Lambda \in \mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$, then so does $I \times \Lambda$.

LEMMA 2. *Let Γ be a closed subset of $\Lambda \in \mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$ and q a nonnegative integer. Let f, g: $\Lambda \times M \to \Lambda \times M$ be Λ-f.p. maps such that $f \mid \Gamma \times M = g \mid \Gamma \times M$ and such that there is a Λ-f.p. homotopy ϕ from $f \mid \Lambda \times M_q$ to $g \mid \Lambda \times M_q$ (rel $\Gamma \times M_q$). Then, ϕ can be extended to a Λ-f.p. homotopy ψ from f to g (rel $\Gamma \times M$). Moreover, if g is a Λ-f.p. homotopy equivalence, then $\hat{\psi}$: $I \times \Lambda \times M \to I \times \Lambda \times M$, defined by $\hat{\psi}(t, \lambda, z) = (t, \psi(t, \lambda, z))$, is an $(I \times \Lambda)$-f.p. homotopy equivalence.*

PROOF. Let $A = (I \times \Lambda \times M_q) \cup (I \times \Gamma \times M) \cup (\{0,1\} \times \Lambda \times M)$ and $\bar{\phi}$: $A \to \Lambda \times M$ be the extension of ϕ defined by

$$\bar{\phi}(t, \lambda, x) = \begin{cases} \phi(t, \lambda, x) & \text{if } x \in M_q, \\ f(\lambda, x)\ (= g(\lambda, x)) & \text{if } \lambda \in \Gamma, \\ f(\lambda, x) & \text{if } t = 0, \\ g(\lambda, x) & \text{if } t = 1. \end{cases}$$

Since M_q is the strong deformation retract of M, we have a homotopy θ: $M \times I \to M$ such that $\theta_0 = \text{id}$, $\theta_1(M) = M_q$ and $\theta_t \mid M_q = \text{id}$ for each $t \in I$. Then, $\bar{\theta} = \text{id} \times \theta$: $I \times \Lambda \times M \times I \to I \times \Lambda \times M$ is an $(I \times \Lambda)$-f.p. homotopy such that $\bar{\theta}_0 = \text{id}$, $\bar{\theta}(A \times I) \subset A$, and $\bar{\theta}_1$: $I \times \Lambda \times M \to I \times \Lambda \times M_q$ is a $(I \times \Lambda)$-f.p. retraction. Observe that $\bar{\phi} = \bar{\phi}\bar{\theta}_0 \mid A$ is Λ-f.p. homotopic to $\bar{\phi}\bar{\theta}_1 \mid A = \phi\bar{\theta}_1 \mid A$ and $\phi\bar{\theta}_1 \mid A$ has the

extension $\phi\bar{\theta}_1$: $I \times \Lambda \times M \to \Lambda \times M$. By [**L**, Lemma 1.1], $\bar{\phi}$ has also a Λ-f.p. extension ψ: $I \times \Lambda \times M \to \Lambda \times M$. This ψ is a wanted extension of ϕ.

Finally, it is easy to prove that $\hat{\psi}$ is $(I \times \Lambda)$-f.p. homotopic to $\mathrm{id}_I \times g$: $I \times \Lambda \times M \to I \times \Lambda \times M$. Therefore, $\hat{\psi}$ is an $(I \times \Lambda)$-f.p. homotopy equivalence if g is a Λ-f.p. homotopy equivalence.

PROOF OF THEOREM 1. Since it is straightforward to verify that the translation map is a homeomorphism of Homeo(M) (or refer to [**H**]), we only need to show that Homeo(M) has the local UV^∞-property at id_M. Let U be a given neighborhood of id_M in Homeo(M) of the form $\bigcap\{[K_i, \Omega_i] \mid i = 1, \dots, p\}$ where K_i is a compact subset of the open subset Ω_i in M for each i.

Definition of V. Without loss of generality, we can assume that $K_1 \cup \cdots \cup K_p \subset M_0$ [**Ha**, Lemma 2.4]. Then, since M_1 is locally convex, there is a finite cover $\mathscr{A}$ of M_0 consisting of convex open sets in M_1 such that

(α) For $E = R$, if $A \in \mathscr{A}$ with $A \cap M_0 \neq \varnothing$, then $A \subset N \times \operatorname{Int} B_2$.

(β) If $A \in \mathscr{A}$ with $A \cap K_i \neq \varnothing$, then its closure $\bar{A} \subset \Omega_i$.

Note that any two $\mathscr{A}$-close maps into M_1 are canonically $\mathscr{A}$-homotopic. Let $\delta > 0$ be a Lebesgue number for the open cover $\{A \cap M_0 \mid A \in \mathscr{A}\}$ of M_0, and let $\mathscr{F}$ be a finite closed cover of M_0 each of whose members has diameter less than δ. Since the union $\bigcup\{\bar{A} \mid A \in \mathscr{A}$ and $\bar{A} \subset \Omega_i\}$ is compact for each $i = 1, \dots, p$, there is a sequence of positive numbers $\varepsilon_2, \varepsilon_3, \dots$ such that for each $i = 1, \dots, p$ if $\bar{A} \subset \Omega_i$, then $A \times E^\infty_{\{\varepsilon_2, \varepsilon_3, \dots\}} \subset \Omega_i$. Now, for each $C \in \mathscr{F}$, since $\operatorname{diam}(C) < \delta$, there is an $A_C \in \mathscr{A}$ such that $C \subset A_C$. Define

$$V = \bigcap\left\{\left[C, A_C \times E^\infty_{\{\varepsilon_2, \varepsilon_3, \dots\}}\right] \mid C \in \mathscr{F}\right\}.$$

(For each $C \in \mathscr{F}$, $A_C \times E^\infty_{\{\varepsilon_2, \varepsilon_3, \dots\}}$ is an open set in M by use of (α).) Observe that V is an open neighborhood of id_M in U: Let $f \in V$. Fix an i ($i = 1, \dots, p$) and let $x \in K_i$. Then, there is a $C \in \mathscr{F}$ such that $x \in C$. So, $A_C \cap K_i \neq \varnothing$; consequently, it follows from (β) that $C \subset A_C \subset \bar{A}_C \subset \Omega_i$; then, $f(x) \in A_C \times E^\infty_{\{\varepsilon_2, \varepsilon_3, \dots\}} \subset \Omega_i$. Therefore, $f(K_i) \subset \Omega_i$ for each i; so, $f \in U$. Moreover, that if $f \in$ Homeo(M), $g \in V$ and $f \mid M_0 = g \mid M_0$, then $f \in V$.

The local strong UV^∞-property. Let $\Lambda \in \mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$ and Γ a closed G_δ-set in Λ. Let g: $\Lambda \to V$ be a continuous map with $g(\Gamma) = \{\mathrm{id}_M\}$. We will prove that g is homotopic rel Γ in V to the constant map Id with $\mathrm{Id}(\Lambda) = \{\mathrm{id}_M\}$.

Let f: $\Lambda \times M \to \Lambda \times M$ be the Λ-f.p. map associated to g, defined by $f_\lambda(z) = g(\lambda)(z)$ for all $(\lambda, z) \in \Lambda \times M$, where $\Lambda \times M$ is the $\mathscr{CG}$-product space as used in [**L**]. Then, f is continuous by [**G**, Theorem 8.17]. By [**D**, Theorem XII.3.1(1)] for the $\mathscr{CG}$-product, we only need to prove that there is a Λ-f.p. isotopy (rel $\Gamma \times M$) F: $I \times \Lambda \times M \to I \times \Lambda \times M$ with $F_0 = f$, $F_1 = \mathrm{id}_{\Lambda \times M}$ and $F_{(t,\lambda)} \in V$ for each $(t, \lambda) \in I \times \Lambda$. We will define by induction a sequence $\{F^n \mid n = 1, 2, \dots\}$ of Λ-f.p. invertible isotopies of $\Lambda \times M$ such that

(0) $F^1_1 = \mathrm{id}_M$ and $F^1_{(t,\lambda)} \in V$ for each $(t, \lambda) \in I \times \Lambda$,

$(1)_n$ $F^n_1 = F^{n-1}_0$ $(n > 1)$,

$(2)_n$ $F^n_0 \mid \Lambda \times M_{n-1} = f \mid \Lambda \times M_{n-1}$ $(n > 0)$,

$(3)_n$ $F^n_t \mid \Lambda \times M_{n-2} = f \mid \Lambda \times M_{n-2}$ for each $t \in I$, $n > 1$, and

$(4)_n$ $F^n_t \mid \Gamma \times M = \mathrm{id}_{\Gamma \times M}$.

Next, define $F\colon I \times \Lambda \times M \to I \times \Lambda \times M$ by

$$F(t, \lambda, z) = \begin{cases} (0, f(\lambda, z)) & \text{if } t = 0, \\ (t, F^n(2^n t - 1, \lambda, z)) & \text{if } 1/2^n \leqslant t \leqslant 1/2^{n-1}. \end{cases}$$

Then, F will be an $(I \times \Lambda)$-f.p. isotopy. It is well defined by $(1)_n$ and continuous by $(2)_n$, $(3)_n$ and $I \times \Lambda \times M = \operatorname{dirlim}(I \times \Lambda \times M_n)$ [**L**, Lemma 0.3]. It also is clear that $F_0 = f$ and that $F_1 = F_1^1 = \mathrm{id}_{\Lambda \times M}$ by (0). Moreover, each $F_{(t,\lambda)} \in V$ by use of (0) and $(3)_n$. Therefore, the proof will be complete. (Note that F is not necessary an invertible isotopy.)

First, let $H\colon I \times \Lambda \times M_0 \to \Lambda \times M$ be the straight-line homotopy from $f \mid \Lambda \times M_0$ to $i_{\Lambda \times M_0}$. Observe that H is a Λ-f.p. homotopy rel $\Gamma \times M_0$ with $H(t \times \lambda \times C) \subset A_C \times E^{\infty}_{\{\varepsilon_2, \varepsilon_3, \ldots\}}$ for each $C \in \mathscr{F}$. Let $\hat{H}\colon I \times \Lambda \times M_0 \to I \times \Lambda \times M$ be the $(I \times \Lambda)$-f.p. map defined by $\hat{H}(t, \lambda, x) = (t, H(t, \lambda, x))$. By [**L**, Lemma 1.2], we can assume that $\hat{H}$ is an $(I \times \Lambda)$-f.p. embedding. Moreover, following the proof of [**L**, Lemma 1.2], to adjust only the $E^{\infty}_{\{\varepsilon_2, \varepsilon_3, \ldots\}}$-component of $\hat{H}$ (rel $\partial I \times \Lambda \times M_0$), we can assume that $\pi_{\Lambda \times M} \hat{H}$ is still a Λ-f.p. homotopy rel $\Gamma \times M_0$ with

$$\pi_M \hat{H}(t \times \lambda \times C) \subset A_C \times E^{\infty}_{\{\varepsilon_2, \varepsilon_3, \ldots\}}$$

for each $C \in \mathscr{F}$ $(*)$. By Lemma 2, $\hat{H}$ has an $(I \times \Lambda)$-f.p. homotopy-equivalence extension $\hat{\psi}^1\colon I \times \Lambda \times M \to I \times \Lambda \times M$ such that

$(a)_1$ $\hat{\psi}^1_1 = \mathrm{id}_{\Lambda \times M}$,

$(b)_1$ $\hat{\psi}^1_0 = f$, and

$(c)_1$ $\hat{\psi}^1_t \mid \Gamma \times M = i_{\Gamma \times M}$ for each $t \in I$.

Therefore, by [**L**, Lemma 3.3], there is an $(I \times \Lambda)$-f.p. homeomorphism F^1 of $I \times \Lambda \times M$ such that F^1 is an extension of $\hat{\psi}^1 \mid (I \times \Lambda \times M_0) \cup (I \times \Gamma \times M) \cup (1 \times \Lambda \times M)$. Now, the properties $(2)_1$ and $(4)_1$ follow from $(b)_1$ and $(c)_1$, respectively. To verify (0), it is clear that $F^1_1 = \mathrm{id}_{\Lambda \times M}$ by $(a)_1$, and that $F^1_{(t,\lambda)} \in V$ by $(*)$ since $\hat{\psi}^1$ is an extension of $\hat{H}$.

Second, consider f and F^1_0. Since $f \mid \Lambda \times M_0 = F^1_0 \mid \Lambda \times M_0$, by use of Lemma 2, we can obtain a Λ-f.p. homotopy $\phi\colon f \mid \Lambda \times M_1 \simeq F^1_0 \mid \Lambda \times M_1$ rel $(\Gamma \times M_1) \cup (\Lambda \times M_0)$. By [**L**, Lemma 1.2], we can assume that $\hat{\phi}\colon I \times \Lambda \times M_1 \to I \times \Lambda \times M$, defined by $\hat{\phi}(t, \lambda, z) = (t, \phi(t, \lambda, z))$, is an $(I \times \Lambda)$-f.p. embedding. By Lemma 2, $\hat{\phi}$ has an $(I \times \Lambda)$-f.p. homotopy equivalence extension $\hat{\psi}^2\colon I \times \Lambda \times M \to I \times \Lambda \times M$ such that

$(a)_2$ $\hat{\psi}^2_1 = F^1_0$,

$(b)_2$ $\hat{\psi}^2_0 = f$, and

$(c)_2$ $\hat{\psi}^2_t \mid \Gamma \times M = i_{\Gamma \times M}$.

Then, by [**L**, Lemma 3.3], there is an $(I \times \Lambda)$-f.p. homeomorphism F^2 of $I \times \Lambda \times M$ such that F^2 is an extension of $\hat{\psi}^2 \mid (I \times \Lambda \times M_1) \cup (I \times \Gamma \times M) \cup (1 \times \Lambda \times M)$. The properties $(1)_2$, $(2)_2$ and $(4)_2$ follow from $(a)_2$, $(b)_2$ and $(c)_2$, respectively. The property $(3)_2$ holds true since $\hat{\psi}^2$ is an extension of $\hat{\phi}$.

Finally, in a similar manner, we can define a wanted $(I \times \Lambda)$-f.p. homeomorphism F^n of $I \times \Lambda \times M$ when an appropriate invertible isotopy F^{n-1} has already been defined. Therefore, the proof is complete. $\square$

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REFERENCES

[D] J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1973.

[G] B. Gray, *Homotopy theory*, Academic Press, New York, 1975.

[H] M. Hale, Doctoral Thesis, Vanderbilt University, 1981.

[Ha] V. L. Hansen, *Some theorems on direct limits of expanding sequences of manifolds*, Math. Scand. **29** (1971), 5–36.

[He] R. E. Heisey, *Manifolds modelled on the direct limit of lines*, Pacific J. Math. **102** (1982), 47–54.

[H-T] R. E. Heisey and H. Toruńczyk, *On the topology of direct limits of* ANR's, Pacific J. Math. **93** (1981), 307–312.

[Hd] J. F. P. Hudson, *Piecewise linear topology*, Math. Lecture Note Ser., Benjamin, New York, 1969.

[L] V. T. Liem, *On R^{∞} (Q^{∞})-manfiold bundles over CW complexes*, Trans. Amer. Math. Soc. (to appear).

[R] T. B. Rushing, *Topological embedding*, Academic Press, New York, 1973.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, UNIVERSITY, ALABAMA 35486