THE SET OF BALANCED ORBITS OF MAPS
OF S^1 AND S^3 ACTIONS

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Abstract. Suppose that the group G = S^1 or G = S^3 acts freely on a space X and on a representation space V for G. Let f: X → V. The paper studies the size of the subset of X consisting of orbits over which the average of f is zero. The result can be viewed as an extension of the Borsuk-Ulam theorem.

1. The average of a map. Let f be a map from S^n to R^n. The classical Borsuk-Ulam theorem says that the set A_f = {x ∈ S^n | f(x) = f(-x)} is nonempty. The formula f(-x) - fx may be viewed as the average of f at the point x, with respect to the antipodal Z_2-actions on the source space S^n, and on the target space R^n. Thus the Borsuk-Ulam theorem can be expressed by saying that for any map f: S^n → R^n there is a point where the average of f (with respect to the antipodal actions) is zero.

The average can be defined for any map of a G-space into a representation space, provided that the transformation group G admits a Haar integral, as is the case for compact groups. A theorem proved by Liulevicius in [5] can be expressed as follows: If G is a nontrivial compact Lie group acting freely on S^m and freely and orthogonally on the unit sphere in a representation space V of dim_R V ≤ m then for any map f: S^m → V there exists an x ∈ S^m where the average of f is zero.

(1.1) Definition. Let X be a G-space and let V be a finite-dimensional representation space of G. Let f: X → V be a (continuous) map. Then the average of f is the map Avf: X → V defined by

\[(Avf)_x = \int g^{-1}f(gx) \, dg.\]

We note the following properties:

(1.2) For any map f: X → V, Avf: X → V is an equivariant map.
(1.3) If f: X → V is equivariant, then Avf = f.

2. The set of balanced points.

(2.1) Definition. Let X be a G-space and let V be a finite-dimensional representation space of G. A map f: X → V is said to be balanced at a point x ∈ X if (Avf)_x = 0. (We will also say then that x is a balanced point of f.) Let A_f denote the set of points of X where f is balanced. Then A_f is an invariant subset of X; it is...
the union of orbits consisting of balanced points. Note that

\[(2.2) \quad A_f = A(A_{f}) = (A(v))^{-1}0.\]

(2.3) Example. Let \( \alpha: X \to X \) be an involution of \( X \) and \( f: X \to \mathbb{R}^n \) be a map of \( X \) into \( \mathbb{R}^n \), with the antipodal involution on \( \mathbb{R}^n \). Then \( A_f = \{ x \in S | f(x) = f(\alpha x) \} \).

Thus the Borsuk-Ulam theorem says that any map \( f: S^n \to \mathbb{R}^n \) is balanced at some point: \( A_f \neq \emptyset \). Various extensions of the Borsuk-Ulam theorem have been concerned with the size of the set \( A_f \) of balanced points of \( f \) for \( \mathbb{Z}_2 \)-actions.

3. The index. A useful invariant of a free involution \( \alpha: X \to X \) on a space \( X \) is its characteristic class, \( u(X) \in H^1(X/\alpha; \mathbb{Z}_2) \); it is the 1st Stiefel-Whitney class of the orbit map \( X \to X/\alpha \), which is a double covering. The index of \( X \), \( \text{Ind}(X) \), is the largest integer \( n \) such that \( u^n(X) \neq 0 \). The index of a free involution was defined by Yang [7] and Conner and Floyd [1]. Fadell, Husseini and Rabinowitz [3, 4] extended the concept of index to actions of compact Lie groups \( G \) other than \( \mathbb{Z}_2 \), including nonfree actions. In this paper we are concerned with the cases \( G = S^1 \) or \( G = S^3 \), i.e., \( G \) is the unit sphere in \( \mathbb{F} \) where \( \mathbb{F} \) is the field of complex numbers, \( \mathbb{C} \), or quaternions, \( \mathbb{H} \). Let \( d \) be the dimension of \( \mathbb{F} \) over \( \mathbb{R} \), that is, \( d = 2 \) for \( \mathbb{F} = \mathbb{C} \), and \( d = 4 \) for \( \mathbb{F} = \mathbb{H} \).

The universal space \( E_G \) for these groups is the infinite dimensional sphere and the classifying space \( E_G/G = B_G \) is the infinite projective space \( P_\infty \mathbb{F} \). The cohomology of \( B_G \) is a polynomial algebra over \( \mathbb{Z} \) on a single generator \( u_\mathbb{F} \in H^d(P_\infty \mathbb{F}) \).

If \( G \) acts freely on a space \( X \), then \( X \) admits an equivariant map \( \phi: X \to E_G \). The characteristic class of the action is \( u_\mathbb{F}(X) = (\phi/G)_*u_\mathbb{F} \in H^d(X/G) \). We define the index, \( \text{Ind}_G(X) \), to be the highest integer \( n \) such that \( u_\mathbb{F}^n(X) \) is of an infinite order in \( H^d(X/G) \). If \( S^d = S^{d(n+1)-1} \) is the unit sphere in \( \mathbb{F}^{n+1} \) with the standard (scalar multiplication) action of \( G \), we will simply write \( u_\mathbb{F}(S^d) = u_\mathbb{F} \). The index of \( S^d \) is \( n \).

The following proposition can be proved in the same way as Proposition 2, part (ii), in Dold [2]:

(3.1) PROPOSITION. If \( S^d \) is the sphere with the standard action and \( \tilde{S}^d \) denotes that sphere with an arbitrary free action, then there exists an equivariant map \( \phi: S^d \to \tilde{S}^d \).

Such a map can be constructed as in [2] because \( P_\infty \mathbb{F} \) is a cell complex whose dimension does not exceed the dimension of the sphere \( \tilde{S}^d \).

(3.2) COROLLARY. \( \text{Ind}_G(\tilde{S}^d) = \text{Ind}_G(S^d) = n \).

PROOF. The inequality \( \text{Ind}_G(\tilde{S}^d) \geq \text{Ind}_G(S^d) \) follows from (3.1). On the other hand, \( \text{Ind}_G(\tilde{S}^d) \) cannot exceed \( n \) since the covering dimension of \( \tilde{S}^d/G \) is at most \( d(n+1)-1 \) as the fibre of the orbit map \( \tilde{S}^d \to \tilde{S}^d/G \) is \( S^{d-1} \), a manifold.

4. Main result. If \( X \) is a \( G \)-space, we will usually denote by \( \overline{X} \) the orbit space \( X/G \). If \( \phi: X \to Y \) is a \( G \)-map from \( X \) to some space \( Y \), \( \overline{\phi} = \phi/G: \overline{X} \to \overline{Y} \) will denote the induced map of the orbit space.

We will be using the Alexander-Spanier cohomology with integer coefficients.

The main result of this paper is the following theorem. It may be viewed as an extension of the theorems of Borsuk-Ulam and Yang to actions of \( S^1 \) and \( S^3 \).
(4.1) **Theorem.** Let \( G = S^1 \) or \( G = S^3 \), respectively, and let \( G \) act freely on a space \( X \) and orthogonally and freely outside the origin on a representation space \( V \) for \( G \) over \( F \). Let \( f: X \to V \) be a map. Then \( \text{Ind}_F(A_f) \geq \text{Ind}_F(X) - \dim_F V \).

By (3.2) we have

(4.2) **Corollary.** If \( \hat{S}_F \) is the unit sphere in \( F^{n+1} \) with any free action of \( G \) and \( f: \hat{S}_F \to V \) is a map of \( \hat{S}_F \) into an orthogonal representation space \( V \) for \( G \) over \( F \), free outside the origin, then \( \text{Ind}_F(A_f) \geq n - \dim_F V \).

(4.3) **Corollary.** The covering dimension of \( A_f \) is at least \( d(n-k)+d-1 \), where \( k = \dim \pi_G \).

This is because \( H^{d(n-k)}(A_f) \neq 0 \) and \( A_f \to A_f \) is a bundle with fibre \( S^{d-1} \).

Actually, a theorem more general than (4.1) will be proved in §6.

5. **Comments on the equivariant cohomology.** In the proof we will be using the equivariant cohomology \( H^*_G \). If \( X \) is a \( G \)-space then \( H^*_G X = H^*(EG \times_G X) \), where \( G \) acts on \( EG \times X \) by \( g(e,x) = (ge, gx) \) and \( EG \times_G X = (EG \times X)/G \). The projection \( EG \times X \to EG \) induces a map \( EG \times_G X \to EG/G = B_G \) which is a bundle with fibre \( X \). If \( G \) acts trivially on \( X \), then \( EG \times X = EG \times X \). If \( G \) acts freely on \( X \), then the projection \( EG \times X \to X \) induces a map \( EG \times_G X \to X/G = \tilde{X} \) which is a bundle with a contractible fibre \( E_G \); in this case \( H^*_G X \equiv H^*X \).

If \( \cdot \) denotes a single point space then \( H^*_G(\cdot) \equiv H^*B_G \); in fact, the constant map \( EG \to \cdot \) induces an isomorphism \( H^*_G(\cdot) \equiv H^*_GEG \equiv H^*B_G \). This ring (in our case of \( G = S^1 \) or \( G = S^3 \) ) is polynomial algebra on a generator \( u_F \in H^d \mathcal{P}_\infty F \).

Let \( V \) be a representation space for \( G \) with \( \dim_F V = m \) and let \( V_0 = V - (0) \).

Since the map \( EG \times_G V \to B_G \) induced by the first projection is an orientable bundle with fibre \( V \), it has its Thom class \( U(V) \in H^m(EG \times_G V, EG \times_G V_0) = H^m_G(V, V_0) \). The restriction of \( U(V) \) to \( V \) will be denoted by \( U'(V) \). The isomorphism \( \pi^*: H^m_BG \cong H^m_GV \) induced by the bundle projection \( \pi: EG \times_G V \to B_G \) maps the Euler class \( e(\pi) \) to \( U'(V) \) : \( U'(V) = \pi^*e(\pi) \). The class \( e(\pi) \) will also be called the Euler class of the action on \( V \) and will be denoted by \( e(\pi) \).

(5.1) **Proposition.** Let \( X \) be a free \( G \)-space, let \( \phi: X \to E_G \) be a classifying map for \( X \) and let \( f: X \to V \) be any equivariant map. Then \( f^*\pi^* = \phi^* \).

**Proof.** Let \( c: E_G \to F^k \) be the constant map to 0. Consider the diagram

\[
\begin{array}{ccc}
E_G \times_G X & \xrightarrow{1 \times_G f} & E_G \times_G V \\
\downarrow 1 \times_G \phi & & \downarrow 1 \times_G c \\
E_G \times_G E_G & \xrightarrow{p} & B_G \\
\end{array}
\]

Since the fibre \( V \) of \( \pi \) is contractible, and the fibre \( E_G \) of \( p \) is contractible, the two triangles are homotopy commutative. Applying the cohomology, we have \( \phi^* = (p(1 \times_G \phi))^* = f^*\pi^* \).
(5.2) Proposition. If $G (= S^1$ or $S^3$) acts on $V = F^k$ by scalar multiplication, then $e(F^k) = u_F^k \in H^{dk} P_k F$.

Proof. In $H^d_G (F^k, F^0) \rightarrow H^d G (F^k, F^0) \rightarrow H^{dk} B_G = H^{dk} P_k F$, the first arrow is an isomorphism since $H^d_G F^k \cong H^{dk} P_k F = 0$ and $H^d_G F^0 \cong H^{dk-1} P_{k-1} F = 0$. It follows that $e(F^k) = \pi^{*-1} U'(F^k) = u_F^k$.

(5.3) Proposition. Let $\tilde{V}$ be an orthogonal representation space for $G$ over $F$, free outside the origin. Then the Euler class $e(\tilde{V}) \neq 0$.

Proof. Let $V = F^k$ be the representation space with the standard (scalar multiplication) action of $G$ and let $S(V) = S_F$ and $S(\tilde{V})$ denote the unit sphere with the corresponding free actions. By (3.1), there is an equivariant map $\phi: S(V) \rightarrow S(\tilde{V})$ which extends to an equivariant map $\psi: V \rightarrow \tilde{V}$. It follows that $\psi^* U'(\tilde{V}) = U'(V)$ which is nonzero by (5.2). Therefore $U'(\tilde{V}) \neq 0$ and $e(\tilde{V}) = \pi^{*-1} U'(\tilde{V}) \neq 0$.

6. Proof of the Theorem. We will prove the following theorem and show that (4.1) is a consequence of it.

(6.1) Theorem. Suppose that $G (= S^1$ or $S^3$, respectively) acts freely on a space $X$ and orthogonally on a representation space $V$ for $G$ over $F$. Let $f: X \rightarrow V$ be a map. If the Euler class $e(V) \neq 0$, then $\text{Ind}_F (A_f) \geq \text{Ind}_F (X) - \dim F V$.

Proof. By (1.2), (1.3) and (2.3) we can assume that $f$ is equivariant; otherwise we can replace $f$ by $Av_f$. Let $\text{Ind}_F (X) = n$ so that $u^n (X)$ is of infinite order and let $k = \dim F V$. We want to show that $u_F^{n-k} (A_f) = u_F^{n-k} (X) | A_f$ is of infinite order. By the continuity of the Alexander-Spanier cohomology, it suffices to show that for every invariant neighborhood $N$ of $A_f$ in $X$, the restriction $u_F^{n-k} (X) | N$ is of infinite order.

The map $f$ can be viewed as an equivariant map of pairs $f: (X, X - A_f) \rightarrow (V, V_0)$. Let $f_N: (N, N - A_f) \rightarrow (V, V_0)$ be the restriction of $f$, let $i$ denote the inclusion $X \rightarrow (X, X - A_f)$ or $V \rightarrow (V, V_0)$ and let $e: (N, N - A_f) \rightarrow (X, X - A_f)$ be the excision map. Then

\[ i^* e^{*-1} (u_F^{n-k} (X) | N) \cup f_N^* (U(V) | (N, N - A_f)) \]

\[ = i^* (u_F^{n-k} (X) \cup f^* U(V)) = u_F^{n-k} (X) \cup f^* i^* U(V) \]

\[ = u_F^{n-k} (X) \cup f^* U'(V) = u_F^{n-k} (X) \cup f^* e(V). \]

Since $H^k B_G = H^k P_k F$ is freely generated by $u_F^k$, $e(V) = mu_F^k$, where $m$ is a nonzero integer since $e(V) \neq 0$.

Now, by (5.1),

\[ u_F^{n-k} (X) \cup f^* e(V) = u_F^{n-k} (X) \cup \phi^* (m u_F^k) \]

\[ = m \cdot u_F^{n-k} (X) \cup \phi^* u_F^k = m \cdot u_F^{n-k} (X) \cup u_F^k (X) \]

is of infinite order.

Finally, Theorem (6.1) and Proposition (5.3) imply Theorem (4.1).
REFERENCES


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