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AN EXAMPLE OF A FAKE s-MANIFOLD WITH A NICE LOCALLY CONTRACTIBLE COMPACTIFICATION

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ABSTRACT. An example is constructed of a topologically complete separable AR X that satisfies the discrete n-cells property for each nonnegative integer n but fails to satisfy the discrete approximation property and be homeomorphic to s even though X arises as the complement of a σ-Z-set in a locally contractible compactum. Such examples are not possible in the setting of ANR compactifications.

1. Introduction. The purpose of this note is to present a simple example of a fake s-manifold that shows that the main result of [**Bow$_1$**] cannot be generalized from the setting of absolute neighborhood retracts (ANR) to the setting of locally contractible spaces. The main result of [**Bow$_1$**] is that a space X satisfying the discrete n-cells property for each nonnegative integer n is equivalent to the space satisfying the discrete approximation property, *provided X arises as the complement of a σ-Z-set in a locally compact separable ANR.* That some extra hypothesis on X is necessary is shown by examples constructed in [**BBMW**] of topologically complete separble ANR's that satisfy the discrete n-cells property for each nonnegative integer n yet fail to satisfy the discrete approximation property and fail to be s-manifolds. We apply the technique developed in [**BBMW**] to construct our example, the starting point of which is Borsuk's construction [**Bor**, **Hu**] of a locally contractible compactum that is not an ANR.

A *map* is a continuous function and id_X denotes the identity map on a space X. A closed subset A of a separable metric space X is a *Z-set* in X, provided for every open cover $\mathcal{U}$ of X there exists a map $\alpha: X \to X - A$ $\mathcal{U}$-close to id_X. The subset A is a *strong-Z-set*, provided, in addition, the map α can be chosen so that the image of α misses a neighborhood of A. If X happens to be locally compact as well, then Z-sets are always strong-Z-sets; however, this is not true in general [**BBMW**]. A countable union of Z-sets is called a *σ-Z-set.* For a nonnegative integer n, a space X is said to satisfy the *discrete n-cells property* if for each countable family of maps $f_i: I^n \to X$, $i = 1, 2, \ldots$, of the n-cell to X and open cover $\mathcal{U}$ of X, there are $\mathcal{U}$-approximations $g_i: I^n \to X$, $i = 1, 2, \ldots$, such that the collection $\{g_i(I^n)\}_{i=1}^{\infty}$ is discrete (each point in X has a neighborhood that meets at most one member of the collection). A space X is said to satisfy the *discrete approximation property* if for each countable family of maps $f_i: I^\infty \to X$, $i = 1, 2, \ldots$, of the Hilbert cube to X and open cover $\mathcal{U}$ of X, there are $\mathcal{U}$-approximations $g_i: I^\infty \to X$, $i = 1, 2, \ldots$, such that the collection $\{g_i(I^\infty)\}_{i=1}^{\infty}$ is discrete. For the importance of the discrete

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properties in the topology of s-manifolds (manifolds modeled on s, the countably infinite product of open intervals $(0,1)$), see [**BBMW**, **To**, **Bow₁**].

I am indebted to Tadeusz Dobrowolski for asking me whether or not the main result of [**Bow₁**] holds in the setting in which he was working, that of locally contractible spaces. Also, I express my sincere appreciation to Doug Curtis for his (as always) helpful advice and suggestions.

2. The example. For a point x in the Hilbert cube $I^\infty = \prod_{i=1}^\infty [0,1]_i$, $x(i)$ denotes the ith coordinate of x. Let $B_\infty = \{x \in I^\infty \mid x(1) = 0\}$, which is a homeomorphic copy of the Hilbert cube, and for each positive integer k let B_k be the k-cube contained in I^∞ that consists of the points $x \in I^\infty$ that satisfy

$$1/(k+1) \le x(1) \le 1/k, \qquad x(i) = 0 \quad \text{for } i > k.$$

Define subspaces C and ∂C of I^∞ as follows:

$$C = B_\infty \cup B_1 \cup B_2 \cup \cdots, \quad \partial C = \partial B_1 \cup \partial B_2 \cup \cdots,$$

where ∂B_k denotes the boundary $(k-1)$-sphere of B_k. The subspace $B_\infty \cup \partial C$ is Borsuk's example of a locally contractible compactum that is not an ANR [**Bor**, **Hu**]. Let D be the following subspace of $I^\infty \times [0,1]$:

$$D = (C \times \{0\}) \cup (\partial C \times [0,1]).$$

D is a topologically complete separable AR (by [**KL** or **Hy**]) and $B = B_\infty \times \{0\}$ is a Z-set in D. In fact, the same arguments used in [**BBMW**, §5] show that there is an instantaneous deformation of D into $D - B$ and that B, though a Z-set, is not a strong-Z-set in D.

Observe that D has a locally contractible compactification, namely $\overline{D} = D \cup (B_\infty \times [0,1])$, and the difference $\overline{D} - D = B_\infty \times (0,1]$ is a σ-Z-set in $\overline{D}$. The latter part of the previous statement follows since there are small retractions of C onto subsets of the form $B_k \cup B_{k-1} \cup \cdots \cup B_1$ that restrict to retractions of $B_\infty \cup \partial C$ onto $(B_k \cup B_{k-1} \cup \cdots \cup B_1) \cap \partial C$.

The example referred to in the Introduction is gotten by taking the *product of D and s reduced about B*, denoted $(D \times s)_B$. $(D \times s)_B$ is the set $[(D - B) \times s] \cup B$ equipped with the topology generated by open subsets of $(D - B) \times s$ and sets of the form $((U - B) \times s) \cup (U \cap B)$, where $U \subset D$ is open. $(D \times s)_B$ is a topologically complete separable AR [**BBMW**, §1], and B is a Z-set in $(D \times s)_B$ but not a strong-Z-set [**BBMW**, Corollary 1.2]. It then follows from [**Bow₂**, Lemma 1, §4] that $(D \times s)_B$ does not have a nice ANR local compactification in the sense that $(D \times s)_B$ does not arise as the complement of a σ-Z-set in a locally compact ANR; however, $(D \times s)_B$ does have a nice locally contractible compactification.

2.1. EXAMPLE. *$(D \times s)_B$ is a topologically complete separable AR that satisfies the discrete n-cells property for each nonnegative integer n but fails to satisfy the discrete approximation property. However, $(D \times s)_B$ has a nice locally contractible compactification in the sense that $(D \times s)_B$ is the complement of a σ-Z-set in a locally contractible compactum.*

For the proof of the claims of the example, we need the following lemma, whose proof involves a straightforward construction and is left as an exercise for the reader.

2.2. LEMMA. *Let α and β be positive integers and $0 < t < 1/2$. Define an α-cell J_t^α contained in the $(\alpha+\beta)$-cell $I^{\alpha+\beta} = [0,1]_1 \times \cdots \times [0,1]_{\alpha+\beta}$ by*

$$J_t^\alpha = [t, 1-t]_1 \times \cdots \times [t, 1-t]_\alpha \times \{1/2\}_{\alpha+1} \times \cdots \times \{1/2\}_{\alpha+\beta}.$$

Then there exists a retraction $r\colon (I^{\alpha+\beta} - J_t^\alpha) \to \partial I^{\alpha+\beta}$ such that r moves the last β coordinates freely while moving the first α coordinates by no more than t. More precisely, let $p_i\colon I^{\alpha+\beta} \to [0,1]_i$ be the ith coordinate projection. Then $|p_i(x) - p_i(r(x))| \leq t$ for each point x in $I^{\alpha+\beta} - J_t^\alpha$ and each $i \in \{1, \ldots, \alpha\}$.

PROOF OF 2.1. First, if $(D \times s)_B$ satisfies the discrete approximation property, then Z-sets are strong-Z-sets [**BBMW**, Proposition 1.3], contradicting the fact that the Z-set B is not a strong-Z-set in $(D \times s)_B$. To show that $(D \times s)_B$ satisfies the discrete n-cells property for each n, it suffices to show that, given a positive number ε and a nonnegative integer n, any countable family of maps $f_1, f_2, \ldots$ of the n-cell I^n into D has 3ε-approximations $g_1, g_2, \ldots$ whose images miss a neighborhood of B and for which $f_i = g_i$ on $f_i^{-1}(D - N_\varepsilon(B))$ for each i, where $N_\varepsilon(B)$ denotes the ε-neighborhood of B in D. Choose a positive integer m so large that, for all $k > m$, $(B_k \times \{0\}) \cup (\partial B_k \times [0, 1/m])$ is contained in $N_\varepsilon(B)$ and, recalling that $D \subset I^\infty \times [0,1]$, so that any move in D that affects only coordinates greater than m moves points at most ε. Since B is a Z-set in D, we may assume that the image of each f_i misses B. Fix a positive integer $\alpha > m$ and let $h\colon B_{\alpha+n+1} \times \{0\} \to I^{\alpha+n+1}$ be the obvious linear homeomorphism induced by the linear homeomorphism $[1/(\alpha+n+2), 1/(\alpha+n+1)] \to [0,1]$ between the first factors. Choose t so small that if r denotes the retraction of Lemma 2.2 with α, $\beta = n+1$, and t, then the distance between $x \in B_{\alpha+n+1} \times \{0\} - h^{-1}(J_t^\alpha)$ and $h^{-1} \circ r \circ h(x)$ is less than 2ε. Since I^n is n-dimensional, we assume that the image of each f_i misses $h^{-1}(J_t^\alpha)$, and by applying $h^{-1} \circ r \circ h$ we obtain 2ε-approximations f_i' to f_i such that $f_i'(I^n) \cap B_{\alpha+n+1} \times \{0\}$ is contained in $\partial B_{\alpha+n+1} \times \{0\}$ for each i. For a positive integer k, let $C_k = B_\infty \cup B_k \cup B_{k+1} \cup \cdots$, and let $\partial C_k = C_k \cap \partial C$. Letting α range over all positive integers greater than m, we obtain 2ε-approximations g_i' to f_i such that $g_i'(I^n) \cap (C_{m+n+2} \times \{0\})$ is contained in $\partial C_{m+n+2} \times \{0\}$. A final move in the $[0,1]$-direction of D produces approximations g_i so that for each i the image of g_i misses $(C_{m+n+3} \times \{0\}) \cup (\partial C_{m+n+3} \times [0, 1/2m])$, a neighborhood of B in D.

We now show that $(D \times s)_B$ has a nice locally contractible compactification. Recall that $\overline{D} = D \cup (B_\infty \times [0,1])$. The reduced product $(\overline{D} \times I^\infty)_B$ contains $(D \times s)_B$ as a dense subspace, and it is easy to show that since $\overline{D}$ is locally contractible, $(\overline{D} \times I^\infty)_B$ is a locally contractible compactum. Since $B_\infty \times (0,1]$ is a σ-Z-set in $\overline{D}$ and $B(I^\infty) = I^\infty - s$ is a σ-Z-set in I^∞, it follows that

$$\begin{aligned}(\overline{D} \times I^\infty)_B - (D \times s)_B &= (\overline{D} - B) \times I^\infty - (D - B) \times s \\ &= ((B_\infty \times (0,1]) \times I^\infty) \cup ((\overline{D} - B) \times B(I^\infty))\end{aligned}$$

is a σ-Z-set in $(\overline{D} \times I^\infty)_B$. The only difficulty is in showing that a set of the form $(B_\infty \times [t,1]) \times I^\infty$ for $0 < t < 1$ is a Z-set in $(\overline{D} \times I^\infty)_B$. For an open neighborhood U of B in $\overline{D}$ let θ be a Urysohn function with $\theta = 0$ on $\overline{D} - U$ and $\theta = 1$ on B, and let H be a contraction of I^∞ to a point with $H_0 = \mathrm{id}_{I^\infty}$ and H_1 constant. Let $p\colon \overline{D} \times I^\infty \to (\overline{D} \times I^\infty)_B$ denote the obvious projection map, and let $r\colon \overline{D} \to \overline{D}$ be a small map for which $(B_\infty \times [0,1]) \cap r(\overline{D}) = \emptyset$. Let $q: \overline{D} \times I^\infty \to \overline{D} \times I^\infty$ be the map defined by $q(d,t) = (r(d), H(t, \theta(d)))$ for $(d,t) \in \overline{D} \times I^\infty$ and observe that since p is a quotient map and $q \circ p^{-1}$ is single valued, $f = p \circ q \circ p^{-1}$ is a well-defined map of $(\overline{D} \times I^\infty)_B$ into $(\overline{D} \times I^\infty)_B$ whose image misses $(B_\infty \times (0,1]) \times I^\infty$. If U is a small enough neighborhood of B and r is close enough to $\mathrm{id}_{\overline{D}}$, then f will be as close to the identity on $(\overline{D} \times I^\infty)_B$ as we wish; hence, $(B_\infty \times [t,1]) \times I^\infty$ is a Z-set in $(\overline{D} \times I^\infty)_B$.

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