AN EXAMPLE OF A FAKE s-MANIFOLD WITH A NICE
LOCALLY CONTRACTIBLE COMPACTIFICATION

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ABSTRACT. An example is constructed of a topologically complete separable
AR X that satisfies the discrete n-cells property for each nonnegative integer n
but fails to satisfy the discrete approximation property and be homeomorphic
to s even though X arises as the complement of a σ-Z-set in a locally con-
tractible compactum. Such examples are not possible in the setting of ANR
compactifications.

1. Introduction. The purpose of this note is to present a simple example of a
fake s-manifold that shows that the main result of [Bow1] cannot be generalized
from the setting of absolute neighborhood retracts (ANR) to the setting of locally
contractible spaces. The main result of [Bow1] is that a space X satisfying the
discrete n-cells property for each nonnegative integer n is equivalent to the space
satisfying the discrete approximation property, provided X arises as the comple-
ment of a σ-Z-set in a locally compact separable ANR. That some extra hypothesis
on X is necessary is shown by examples constructed in [BBMW] of topologically
complete separable ANR's that satisfy the discrete n-cells property for each nonneg-
ative integer n yet fail to satisfy the discrete approximation property and fail to
be s-manifolds. We apply the technique developed in [BBMW] to construct our
example, the starting point of which is Borsuk's construction [Bor, Hu] of a locally
contractible compactum that is not an ANR.

A map is a continuous function and idX denotes the identity map on a space X.
A closed subset A of a separable metric space X is a Z-set in X, provided for every
open cover U of X there exists a map α: X → X − A close to idX. The subset A
is a strong-Z-set, provided, in addition, the map α can be chosen so that the image
of α misses a neighborhood of A. If X happens to be locally compact as well, then
Z-sets are always strong-Z-sets; however, this is not true in general [BBMW].
A countable union of Z-sets is called a σ-Z-set. For a nonnegative integer n, a
space X is said to satisfy the discrete n-cells property if for each countable family
of maps fi: In → X, i = 1, 2, ..., of the n-cell to X and open cover U of X, there
are U-approximations gi: In → X, i = 1, 2, ..., such that the collection {gi(In)}i=1
is discrete (each point in X has a neighborhood that meets at most one member
of the collection). A space X is said to satisfy the discrete approximation property
if for each countable family of maps fi: I∞ → X, i = 1, 2, ..., of the Hilbert cube
to X and open cover U of X, there are U-approximations gi: I∞ → X, i = 1, 2, ...,;
such that the collection {gi(I∞)}i=1 is discrete. For the importance of the discrete

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properties in the topology of s-manifolds (manifolds modeled on s, the countably
infinite product of open intervals (0, 1)), see [BBMW, To, Bowi].

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result of [Bowi] holds in the setting in which he was working, that of locally
contractible spaces. Also, I express my sincere appreciation to Doug Curtis for his
(as always) helpful advice and suggestions.

2. The example. For a point x in the Hilbert cube $I^\infty = \prod_{i=1}^{\infty}[0, 1]$, $x(i)$
denotes the i'th coordinate of x. Let $B_\infty = \{x \in I^\infty \mid x(1) = 0\}$, which is a
homeomorphic copy of the Hilbert cube, and for each positive integer k let $B_k$ be
the k-cube contained in $I^\infty$ that consists of the points $x \in I^\infty$ that satisfy
\[ \frac{1}{k+1} \leq x(1) \leq \frac{1}{k}, \quad x(i) = 0 \quad \text{for } i > k. \]

Define subspaces C and $\partial C$ of $I^\infty$ as follows:
\[ C = B_\infty \cup B_1 \cup B_2 \cup \cdots, \quad \partial C = \partial B_1 \cup \partial B_2 \cup \cdots, \]
where $\partial B_k$ denotes the boundary $(k - 1)$-sphere of $B_k$. The subspace $B_\infty \cup \partial C$
is Borsuk's example of a locally contractible compactum that is not an ANR
[Bor, Hu]. Let $D$ be the following subspace of $I^\infty \times [0, 1]$
\[ D = (C \times \{0\}) \cup (\partial C \times [0, 1]). \]

$D$ is a topologically complete separable AR (by [KL or Hy]) and $B = B_\infty \times \{0\}$
is a Z-set in $D$. In fact, the same arguments used in [BBMW, §5] show that there
is an instantaneous deformation of $D$ into $D - B$ and that $B$, though a Z-set,

\[ \text{Observe that } D \text{ has a locally contractible compactification, namely } \bar{D} = D \cup
(B_\infty \times [0, 1]), \text{ and the difference } \bar{D} - D = B_\infty \times (0, 1) \text{ is a } \sigma\text{-Z-set in } \bar{D}. \text{ The latter part of the previous statement follows since there are small retractions of } C \text{ onto
subsets of the form } B_k \cup B_{k-1} \cup \cdots \cup B_1 \text{ that restrict to retractions of } B_\infty \cup \partial C \text{ onto } (B_k \cup B_{k-1} \cup \cdots \cup B_1) \cap \partial C. \]

The example referred to in the Introduction is gotten by taking the product of
$D$ and s reduced about $B$, denoted $(D \times s)_B$. $(D \times s)_B$ is the set $[(D - B) \times s] \cup B$
equipped with the topology generated by open subsets of $(D - B) \times s$ and sets of the form $((U - B) \times s) \cup (U \cap B)$, where $U \subset D$ is open. $(D \times s)_B$ is a topologically
complete separable AR [BBMW, §1], and $B$ is a Z-set in $(D \times s)_B$ but not a
strong-$\sigma$-set [BBMW, Corollary 1.2]. It then follows from [Bow2, Lemma 1, §4]
that $(D \times s)_B$ does not have a nice ANR local compactification in the sense that
$(D \times s)_B$ does not arise as the complement of a $\sigma$-$\sigma$-set in a locally compact ANR;

\[ \text{however, } (D \times s)_B \text{ does have a nice locally contractible compactification.} \]

For the proof of the claims of the example, we need the following lemma, whose
proof involves a straightforward construction and is left as an exercise for the reader.
2.2. **Lemma.** Let \( \alpha \) and \( \beta \) be positive integers and \( 0 < t < 1/2 \). Define an \( \alpha \)-cell \( J^\alpha \) contained in the \((\alpha + \beta)\)-cell \( I^{\alpha + \beta} = [0,1]_1 \times \cdots \times [0,1]_{\alpha + \beta} \) by

\[
J^\alpha_t = [t,1-t]_1 \times \cdots \times [t,1-t]_{\alpha} \times \{1/2\}_{\alpha+1} \times \cdots \times \{1/2\}_{\alpha+\beta}.
\]

Then there exists a retraction \( r: (I^{\alpha + \beta} - J^\alpha_t) \to \partial I^{\alpha + \beta} \) such that \( r \) moves the last \( \beta \) coordinates freely while moving the first \( \alpha \) coordinates by no more than \( t \). More precisely, let \( p_i: I^{\alpha + \beta} \to [0,1]_i \) be the \( i \)th coordinate projection. Then \( |p_i(x) - p_i(r(x))| \leq t \) for each point \( x \) in \( I^{\alpha + \beta} - J^\alpha_t \) and each \( i \in \{1, \ldots, \alpha\} \).

**Proof of 2.1.** First, if \((D \times s)_B\) satisfies the discrete approximation property, then \( Z \)-sets are strong-\( Z \)-sets [BBMW, Proposition 1.3], contradicting the fact that the \( Z \)-set \( B \) is not a strong-\( Z \)-set in \((D \times s)_B\). To show that \((D \times s)_B\) satisfies the discrete \( n \)-cells property for each \( n \), it suffices to show that, given a positive number \( \varepsilon \) and a nonnegative integer \( n \), any countable family of maps \( f_1, f_2, \ldots \) of the \( n \)-cell \( I^n \) into \( D \) has \( 3\varepsilon \)-approximations \( g_1, g_2, \ldots \) whose images miss a neighborhood of \( B \) and for which \( f_i = g_i \) on \( f_i^{-1}(D - \varepsilon(B)) \) for each \( i \), where \( \varepsilon(B) \) denotes the \( \varepsilon \)-neighborhood of \( B \) in \( D \). Choose a positive integer \( m \) so large that, for all \( k > m \), \((B_k \times \{0\}) \cup (\partial B_k \times [0,1/m])\) is contained in \( \varepsilon(B) \) and, recalling that \( D \subset I^\infty \times [0,1] \), so that any move in \( D \) that affects only coordinates greater than \( m \) moves points at most \( \varepsilon \). Since \( B \) is a \( Z \)-set in \( D \), we may assume that the image of each \( f_i \) misses \( B \). Fix a positive integer \( \alpha > m \) and let \( h: B_{\alpha + n + 1} \times \{0\} \to I^{\alpha + n + 1} \) be the obvious linear homeomorphism induced by the linear homeomorphism \([1/(\alpha + n + 2), 1/(\alpha + n + 1)] \to [0,1]\) between the first factors. Choose \( t \) so small that if \( r \) denotes the retraction of \( \alpha \)-cell \( J^\alpha_t \), then \( d(h^{-1} \circ r \circ h(x)) \leq 2\varepsilon \). Since \( I^n \) is \( n \)-dimensional, we assume that the image of each \( f_i \) misses \( h^{-1}(J^\alpha_t) \), and by applying \( h^{-1} \circ r \circ h \) we obtain \( 2\varepsilon \)-approximations \( f'_i \) to \( f_i \) such that \( f'_i(I^n) \cap B_{\alpha + n + 1} \times \{0\} \) is contained in \( \partial B_{\alpha + n + 1} \times \{0\} \) for each \( i \). For a positive integer \( k \), let \( C_k = B_\infty \cup B_k \cup B_{k+1} \cup \cdots \), and let \( \partial C_k = C_k \cup \partial D \). Letting \( \alpha \) range over all positive integers greater than \( m \), we obtain \( 2\varepsilon \)-approximations \( g'_i \) to \( f'_i \) such that \( g'_i(I^n) \cap (C_{m+n+3} \times \{0\}) \) is contained in \( \partial C_{m+n+3} \times \{0\} \). A final move in the \([0,1]\)-direction of \( D \) produces approximations \( g_i \) so that for each \( i \), the image of \( g_i \) misses \( (C_{m+n+3} \times \{0\}) \cup (\partial C_{m+n+3} \times [0,1/2m]) \), a neighborhood of \( B \) in \( D \).

We now show that \((D \times s)_B\) has a nice locally contractible compactification. Recall that \( D = D \cup (B_\infty \times [0,1]) \). The reduced product \((\overline{D} \times I^\infty)_B\) contains \((D \times s)_B\) as a dense subspace, and it is easy to show that since \( \overline{D} \) is locally contractible, \((\overline{D} \times I^\infty)_B \) is a locally contractible compactum. Since \( B_\infty \times (0,1] \) is a \( \sigma \)-\( Z \)-set in \( \overline{D} \) and \( B(I^\infty) = I^\infty - s \) is a \( \sigma \)-\( Z \)-set in \( I^\infty \), it follows that

\[
(\overline{D} \times I^\infty)_B - (D \times s)_B = (\overline{D} - B) \times I^\infty - (D - B) \times s
= (B_\infty \times (0,1]) \times I^\infty \cup ((\overline{D} - B) \times B(\infty))
\]

is a \( \sigma \)-\( Z \)-set in \((\overline{D} \times I^\infty)_B \). The only difficulty is in showing that a set of the form \((B_\infty \times [t,1]) \times I^\infty \) for \( 0 < t < 1 \) is a \( Z \)-set in \((\overline{D} \times I^\infty)_B \). For an open neighborhood \( U \) of \( B \) in \( \overline{D} \), let \( \theta \) be a Urysohn function with \( \theta = 0 \) on \( \overline{D} - U \) and \( \theta = 1 \) on \( B \), and let \( H \) be a contraction of \( I^\infty \) to a point with \( H_0 = \text{id}_{I^\infty} \) and \( H_1 \) constant. Let \( p: \overline{D} \times I^\infty \to (\overline{D} \times I^\infty)_B \) denote the obvious projection map, and let \( r: \overline{D} \to \overline{D} \) be a
small map for which \((B_\infty \times [0, 1]) \cap r(D) = \emptyset\). Let \(q: D \times I^\infty \to D \times I^\infty\) be the map defined by \(q(d, t) = (r(d), H(t, \theta(d)))\) for \((d, t) \in D \times I^\infty\) and observe that since \(p\) is a quotient map and \(q \circ p^{-1}\) is single valued, \(f = p \circ q \circ p^{-1}\) is a well-defined map of \((D \times I^\infty)_B\) into \((D \times I^\infty)_B\) whose image misses \((B_\infty \times (0, 1]) \times I^\infty\). If \(U\) is a small enough neighborhood of \(B\) and \(r\) is close enough to \(id_B\), then \(f\) will be as close to the identity on \((D \times I^\infty)_B\) as we wish; hence, \((B_\infty \times [t, 1]) \times I^\infty\) is a \(Z\)-set in \((D \times I^\infty)_B\).

**REFERENCES**


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