A NOTE ON TWIST SPUN KNOTS

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ABSTRACT. A movie presentation for the twist spun knots of an arc is given.

Some time ago Francisco González-Acuña asked me for a movie presentation of the twist spun knots defined and studied by Zeeman [Z]. Since then, other low dimensional topologists have posed the same question to me. Perhaps the following easy solution may have some interest.

**LEMMA.** Let $K$ be the knot in plat presentation of Figure 1 where $x$ belongs to the braid group $B_{2m+1}$. Then the $n$-twist spun knot of $K$ is given by the diagram of Figure 2.

**REMARK.** In Figure 2 we use Lomonaco’s notation [L]. The diagram of Figure 3(a) is explained in Figure 3(b). The critical saddle point occurs at level $t = \theta$.

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**Figure 1.**

**Figure 2.**

**Figure 3.**

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PROOF. By deleting the interior of a regular neighborhood of a point in \( K \) we obtain a pair \((B^3, B^3 \cap K)\) which we denote by \((\tilde{B}^3, \tilde{K})\). Consider the height function \( f: B^3 \to [-1,1] \) of Figure 4.

Let \( t \in [0,1] \) be the parameter measuring the spinning process of \( B^3 \) so that at time \( t = 1 \) the ball arrives to its original position. Assume that the twisting of \( \tilde{K} \) occurs during the interval \([\frac{1}{2}, 1]\).

We want a movie of the \( n \)-twist spun knot \( \tilde{K}_n \) of \( K \) with respect to "hyperplane" sections \( S^3_r, r \in (-1,1) \), where \( S^3_r \) is the result of spinning the subset \( f^{-1}(r) \) of \( B^3 \). For \( r \in \{-1,1\}, f^{-1}(r) \) is just a point.

To achieve this we first define an isotopy of \( S^4 \) which places the saddle points of \((f \times id)|\tilde{K}_n\) in the level \( S^3_0 \). This isotopy is defined in three steps.

Step 1. Consider the model halfball \( D^3 \) and the set \( A \) of \( m + 1 \) arcs shown in Figure 5(a). There is an isotopy \( g: A \times I \to D^3 \) which pushes \( m \) arcs of \( A \) onto the boundary. In Figure 5(b) we see the images of \( A \) for some values of the parameter.
Step 2. Let $B_+^3$ and $B_-^3$ be the halfballs $f^{-1}[0,1]$ and $f^{-1}[0,-1]$ of $B^3$. Let $F_+$ and $F_-$ be homeomorphisms $F_{\pm} : (D^3, A) \to (B_+^3, B_-^3 \cap \bar{K})$ and define isotopies $g_{\pm} : \bar{K} \times I \to B^3$ as follows: $g_{\pm}$ is the identity map in $(B_+^3 \cap \bar{K}) \times t$, $t \in I$, and equals $F_{\pm}g_{\pm}^{-1}$ in $(B_-^3 \cap \bar{K}) \times I$. We embed $g_{\pm}$ in ambient isotopies $G_{\pm} : B^3 \times I \to B^3$. Note that $G_-((B_-^3 \cap \bar{K}) \times 1)$ is the set of arcs $b$ together with the point $c$ of Figure 6, if we think of $F_-$ as the identity map. Under this condition the set $G_+((B_+^3 \cap \bar{K}) \times 1)$
is the image of \( b \cup c \) under the action of \( x^{-1} \in B_{2m+1} \) on \( (f^{-1}(0), f^{-1}(0) \cap \bar{K}) \). In Figure 7 we show the case \( x = \sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1} \).

**Step 3.** We now define an isotopy of \( S^4 \). This isotopy connects the identity map with a map \( h: S^4 \rightarrow S^4 \) defined as follows. The map \( h \) realizes \( G_+ \) when we spin \( B^3 \) between \( t = 0 \) and \( t = 1/12 \), it is constant for \( t \in \left[ \frac{1}{12}, \frac{3}{12} \right] \), and undoes \( G_+ \) between \( t = 2/12 \) and \( t = 3/12 \). After that, \( h \) does \( G_- \) in \( \left[ \frac{3}{12}, \frac{4}{12} \right] \), is constant in \( \left[ \frac{4}{12}, \frac{5}{12} \right] \) and undoes \( G_- \) in \( \left[ \frac{5}{12}, \frac{6}{12} \right] \). During \( [\frac{1}{2}, 1] \) \( h \) is the identity map. In Figure 8 we see \( h((B^3 \cap \bar{K}) \times \left[ \frac{3}{12}, \frac{6}{12} \right]) \).

The knot \( h(K_n) \) is ambient isotopic to \( K_n \) but all its saddle points with respect to \( f \times \text{id} \) are at level \( S^3_{0} \). We only need to understand \( S^3_{0} \cap h(K_n) \). The pair \((S^3_{0}, S^3_{0} \cap h(K_n))\) is the union of the result of spinning \( (f^{-1}(0), f^{-1}(0) \cap \bar{K}) \) during \( t \in [0, \frac{1}{2}] \), with the result of spinning \( x^{-1}(b \cup c) \) during \( t \in \left[ \frac{1}{12}, \frac{2}{12} \right] \), with the result of spinning \( b \cup c \) during \( t \in \left[ \frac{4}{12}, \frac{5}{12} \right] \), with the result of \( n \)-twist spinning \( (f^{-1}(0), f^{-1}(0) \cap \bar{K}) \) during \( t \in \left[ \frac{1}{2}, 1 \right] \). The picture for \( \bar{K} \) given by \( x = \sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1} \) is in Figure 9.

![Figure 9](image)
We have that \((S^3_0, S^3_0 \cap h(K_n))\) is a torus link with \(2m + 1\) trivial components and \(n\)-full twists, together with two sets of bands which correspond to the saddle points. By shrinking the bands with middle lines \(x^{-1}(b \cup c)\) suitably we see that Figure 9 becomes Figure 10.

**COROLLARY.** The torus link \(\{(2m + 1)n, 2m + 1\}\) is a slice of a trivial knot in \(S^4\). Links of the form depicted in Figure 11(b) have the same property.

**PROOF.** For the first part take \(x \in B_{2m+1}\) such that \(K\) is a trivial knot. For the second part, remember that the 1-twist spun knot of \(K\) is trivial \([Z]\).

**REFERENCES**


(a) \(K\) is a knot
(b) \(-K\) is the mirror image

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