ON A QUESTION OF FAITH
IN COMMUTATIVE ENDOMORPHISM RINGS

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ABSTRACT. Given a commutative ring $R$, let $Q(R)$ denote its maximal ring of quotients and, for any ideal $I$ of $R$, let $\operatorname{End}(I)$ denote the ring of $R$-endomorphisms of $I$. It is known that if $Q(R)$ is a self-injective ring then $\operatorname{End}(I)$ is commutative for each ideal $I$ of $R$. Carl Faith has asked if the converse holds. It does if $R$ is either Noetherian or has no nontrivial nilpotent elements but here we produce an example to show that it does not hold in general.

Introduction. Throughout this paper $R$ is a commutative ring with identity and $Q(R)$ will denote its maximal ring of quotients. The definition and fundamental properties of $Q(R)$ are to be found in, for example, the books of Lambek [7] and Stenström [9]. In particular, Theorem 4.1 on p. 279 of [9] shows that $Q(R)$ is a self-injective ring if and only if for every ideal $I$ of $R$ and every $R$-homomorphism $\phi: I \to R$ there exists a faithful ideal $J$ of $R$ and an $R$-homomorphism $\psi: J \to R$ such that $I \subseteq J$ and $\psi$ extends $\phi$.

For every ideal $I$ of $R$ we let $\operatorname{End}(I)$ denote the ring of $R$-endomorphisms of $I$. In Proposition 1.2 of [3], Cox showed that $\operatorname{End}(J)$ is commutative whenever $J$ is a faithful ideal. It then follows easily from the previous paragraph that if $Q(R)$ is a self-injective ring then $\operatorname{End}(I)$ is commutative for every ideal $I$ of $R$. Carl Faith has asked, [4, p. 199, problem 10] and [5, p. 98, problem 6], if the converse holds. Alamelu, [1 and 2], and Cox [3] have shown that it does with some restrictions on $R$, in particular, if $R$ is either Noetherian or has no nontrivial nilpotent elements.

We produce below an example of a ring $R$ which gives a negative answer to Faith’s general question.

The example. Let $A$ be a discrete valuation ring with maximal ideal $At$ and quotient field $K$. Moreover, assume that $A$ is not complete, for example $A$ can be taken to be countable. Let $M$ denote the $A$-module $K/At$ and for each $n \in \mathbb{N}$ let $M_n$ denote the $A$-submodule $At^{-n}/A$ of $M$ where $At^{-n} = \{at^{-n}: a \in A\} = \{ut^k: u \text{ is a unit in } A, k \in \mathbb{Z}, k \geq -n\}$. Then $M$ is the direct union $\bigcup_{n \in \mathbb{N}} M_n$ and any proper nonzero $A$-submodule of $M$ is $M_n$ for some $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ let $E_n$ denote the $A$-endomorphism ring of $M_n$ and let $E$ denote the $A$-endomorphism ring of $M$. Any $f$ in $E_n$ is determined by $f(t^{-n} + A)$, which is of the form $a(t^{-n} + A)$ for some $a$ in $A$. This produces a ring isomorphism $\phi: E_n \to A/At^n$ given by $\phi(f) = a + At^n$. 

Received by the editors April 30, 1985 and, in revised form, October 2, 1985.


Key words and phrases. Endomorphism ring, maximal quotient ring, self-injective, trivial extension.
If $x$ is an arbitrary nonzero element of some $M_n$ we can write $x = ut^{m-n} + A$ where $u$ is a unit in $A$ and $m < n$. Then if $g: M_n \to M$ is an $A$-homomorphism with $g(x) = vt^{-k} + A$, where $v$ is a unit and $k \geq 0$, we have

$$0 = g(t^{n-m}x) = t^{n-m}g(x) = vt^{n-m-k} + A$$

and so $-k \geq m - n$. Hence $g(x) \in M_{n-m} \subseteq M_n$. This argument allows us to define, for any $m, n \in \mathbb{N}$ with $m \leq n$, the restriction functions $\phi_{mn}: E_m \to E_n$ by $\phi_{mn}(g) = g|_{M_n}$ and $\phi_n: E \to E_n$ defined similarly. This gives a projective system $(E_m, \phi_{mn})_{m,n \in \mathbb{N}}$ of rings and ring homomorphisms and it is readily checked that

$$E = \text{proj lim}. E_m.$$ 

Since, from above, each $E_n$ is isomorphic to the commutative ring $A/At^n$, it follows that $E$ is a commutative ring (see, for example, Proposition 1.9 on p. 53 of Lafon [6]).

We now let $R$ be the trivial extension of $A$ by $M$. In other words, $R = A \oplus M$, made into a ring by defining addition componentwise and multiplication by

$$(a_1 + m_1)(a_2 + m_2) = a_1a_2 + a_1m_2 + a_2m_1$$

for all $a_1, a_2 \in A$, $m_1, m_2 \in M$. This procedure was called the principle of idealization by Nagata [8] since the ring $R$ contains the ring $A$ and $M$ is an ideal of $R$, with $M^2 = 0$. We determine the ideals of $R$.

First, if $S$ is any $A$-submodule of $M$, then, by our multiplication in $R$, $S$ is an ideal of $R$. Moreover, any ideal of $R$ contained in $M$ is such a submodule. Thus the ideals of $R$ contained in $M$ are precisely $0$, $M$ and the $M_n$. Furthermore, each $M_n$ is just the principal ideal of $R$ generated by $t^{n} + A$. On the other hand, if $r$ is an element of $R$ not in $M$, say $r = ut^n + m$, where $u$ is a unit in $A$, $n \geq 0$ and $m \in M$, then by the divisibility of the $A$-module $M = K/A$ we have $t^nM = M$. Using this, it is not difficult to show that the principal ideal $Rr$ is $At^n \oplus M$ and coincides with $Rt^n$. In particular, we have that all ideals of $R$ not contained in $M$ are principal. Thus all the ideals of $R$ except $M$ are principal.

Now from the definition of $R$ it is easily shown that the $R$-endomorphism ring of $M$ coincides with its $A$-endomorphism ring, in other words $\text{End}_R(M) = E$. In particular, from above, the $R$-ideal $M$ has a commutative endomorphism ring. Since all the other ideals of $R$ are principal, they also have commutative endomorphism rings. Thus to establish our example it remains to show that $Q(R)$ is not self-injective.

In fact, $Q(R) = R$. To see this we first note that for any ideal $I$ of $R$ contained in $M$ we have $MI = 0$, while if $J$ is a proper ideal of $R$ not contained in $M$ then $J = At^n \oplus M$ for some $n > 0$ and so $M_nJ = 0$. Thus the only faithful ideal in $R$ is $R$ itself. That $Q(R) = R$ now follows from Proposition 4 on p. 39 of Lambek [7].

Now let $\hat{A}$ denote the completion of $A$ and choose an element $b$ belonging to $\hat{A}$ but not to $A$. Then $b$ can be represented by a formal power series

$$b = u_0 + u_1 + \cdots + u_n t^n + \cdots$$

where the $u_i$ are elements of $A$ coming from a fixed set of representatives for $A$ modulo $At$ and, since $b$ is not in $A$, $u_i$ is nonzero for infinitely many $i$. Then multiplication by $b$ gives an $R$-endomorphism $h$ of the ideal $M$ which differs from any multiplication of $M$ by an element from $A$. However, for any $R$-endomorphism
k: R → R, if k(1) = a + m where a ∈ A, m ∈ M, then the restriction of k to M is just multiplication by a. This shows that our homomorphism h: M → R does not extend to R. Hence R, and so Q(R), is not self-injective.

ACKNOWLEDGEMENT. It is a pleasure to acknowledge the referee for suggestions that improved the presentation.

REFERENCES

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