

A DIFFERENTIAL EQUATION THEORETIC INTERPRETATION OF A GEOMETRIC RESULT OF HARTOGS

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ABSTRACT. A result of Hartogs on the location of singularities of holomorphic functions can be neatly proved by the theory of overdetermined systems of linear differential equations.

The pioneering work of Bochner [1] and its substantial generalization by Ehrenpreis [2] tell us that the Hartogs phenomenon concerning removable singularities of holomorphic functions of several variables are most neatly handled by the theory of overdetermined systems of linear differential equations with constant coefficients. The results of Ehrenpreis [2] and Ehrenpreis [3, Chapter XI, §2] were extended by Kawai [6] to the general situation where the equations are with variable coefficients and the boundary in question is not necessarily a linear variety. The purpose of this note is to notice that a result in Kawai [6] gives a neat proof of another interesting result (the theorem below) of Hartogs on the location of the singularities of a holomorphic function of several complex variables. (See Hartogs [4] for the original statement of the result. See also Hitotumatu [5, Chapter 3, §5]. The original result is more precise than ours in that Hartogs does not assume any regularity of the exceptional set E in the theorem below.) The author hopes that the interpretation of the Hartogs theorem in the light of the theory of general overdetermined systems as is given here will help us to acquire better understanding of the nature of a domain of existence for a system of linear differential equations, a challenging and intriguing problem posed by Ehrenpreis [3, p. 338].

THEOREM (HARTOGS [4]). *Let Ω be an open subset of \mathbb{C}^n ($n \geq 1$) and let $\varphi(z, \bar{z})$ be a (complex-valued) real analytic function defined on Ω . Let E denote $\{(z, w) \in \Omega \times \mathbb{C}; w = \varphi(z, \bar{z})\}$. Suppose that $\Omega \times \mathbb{C} - E$ is locally a domain of holomorphy near E , that is, suppose that for each point p in E we can find an open neighborhood U of p and a holomorphic function $h(z, w)$ defined on $U - E$ such that h cannot be extended across E . Then $\varphi(z, \bar{z})$ is holomorphic.*

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PROOF. Let s and x_j (resp., t and y_j) denote the real part (resp., the imaginary part) of w and z_j ($j = 1, \dots, n$), respectively. Let $f(x, y)$ and $g(x, y)$ respectively denote the real part and the imaginary part of $\varphi(z, \bar{z})$. Let us now regard $\Omega \times \mathbb{C}$ (resp., E) as a $2(n + 1)$ -dimensional (resp., $2n$ -dimensional) real analytic manifold, and we denote by X (resp., Y) its complexification. Let \mathcal{M} denote the Cauchy-Riemann system on $\Omega \times \mathbb{C}$, that is,

$$(1) \quad \mathcal{M}: \begin{cases} \frac{1}{2}(\partial/\partial x_j + \sqrt{-1} \partial/\partial y_j)u = 0 & (j = 1, \dots, n), \\ \frac{1}{2}(\partial/\partial s + \sqrt{-1} \partial/\partial t)u = 0. \end{cases}$$

We may, of course, regard \mathcal{M} as a system of linear differential equations defined on X , and we denote by V its characteristic variety, a subvariety of T^*X .

Since a holomorphic function on $\Omega \times \mathbb{C} - E$ is by definition a (hyperfunction) solution u of (1) defined on $\Omega \times \mathbb{C} - E$, Theorem 1 of Kawai [6] implies that E should be characteristic with respect to \mathcal{M} ; otherwise u should be extended across E . It then follows from the definition of the characteristicness (Sato-Kawai-Kashiwara [7, Chapter II, Definition 3.5.4]) that the conormal bundle T_Y^*X has nontrivial intersection with V over E . This means, by definition, that for each point $p = (q, \varphi(q))$ in E we can find complex numbers $(\alpha, \beta) (\neq (0, 0))$ such that

$$\begin{aligned} & (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n; \sigma, \tau) \\ & = \alpha \left(-\frac{\partial f}{\partial x_1}(q), \dots, -\frac{\partial f}{\partial x_n}(q), -\frac{\partial f}{\partial y_1}(q), \dots, -\frac{\partial f}{\partial y_n}(q); 1, 0 \right) \\ & + \beta \left(-\frac{\partial g}{\partial x_1}(q), \dots, -\frac{\partial g}{\partial x_n}(q), -\frac{\partial g}{\partial y_1}(q), \dots, -\frac{\partial g}{\partial y_n}(q); 0, 1 \right) \end{aligned}$$

belongs to

$$V = \left\{ (x, y, s, t; \xi, \eta, \sigma, \tau) \in T^*X; \xi_j + \sqrt{-1} \eta_j = 0 \ (j = 1, \dots, n) \right. \\ \left. \text{and } \sigma + \sqrt{-1} \tau = 0 \right\}.$$

Since $(\alpha, \beta) \neq (0, 0)$, we may suppose $\alpha \neq 0$ and then, by the homogeneity, we may further assume $\alpha = 1$. (If $\beta \neq 0$, we may normalize β to be 1.) After this normalization we find

$$(2) \quad 1 + \sqrt{-1} \beta = 0,$$

that is, $\beta = \sqrt{-1}$. Hence we have

$$(3) \quad \left(-\frac{\partial f}{\partial x_j}(q) - \sqrt{-1} \frac{\partial g}{\partial x_j}(q) \right) + \sqrt{-1} \left(-\frac{\partial f}{\partial y_j}(q) - \sqrt{-1} \frac{\partial g}{\partial y_j}(q) \right) = 0 \\ (j = 1, \dots, n),$$

that is,

$$(4) \quad \left(\frac{\partial}{\partial x_j} + \sqrt{-1} \frac{\partial}{\partial y_j} \right) (f + \sqrt{-1} g) = 0 \quad (j = 1, \dots, n)$$

holds identically. This means that $\varphi = f + \sqrt{-1} g$ is holomorphic, completing the proof of the theorem.

REMARK. If the (real) codimension of the exceptional set E in the theorem is equal to or greater than 4, it is always a removable singularity for holomorphic functions. Hence the situation considered in the theorem is the only meaningful one in our context.

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