THE K-FUNCTIONAL FOR H^P AND BMO IN THE POLY-DISK

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Abstract. Peetre's K-functional for the Hardy space H^p, 0 < p < +∞, and the space BMO of functions of bounded mean oscillation is explicitly characterized in the case of a product of upper half-spaces.

1. Introduction. In this note we consider the K-functional for the Hardy spaces H^p, 0 < p < +∞, and the space BMO of functions of bounded mean oscillation in the case of a product of upper half-spaces. The main result in [J] is a characterization of the K-functional for H^p and BMO in R^n in terms of a certain truncated square function, and here we prove the analogous result in the product case. Our proof is based on a refinement of some ideas in Chang-Fefferman [C-F] and uses a Calderón-Zygmund type procedure where certain families of open sets play the role of the dyadic cubes in the classical case.

Let us briefly recall some relevant facts and definitions (see [C-F and B-L] for more details):

In what follows we shall, for simplicity, work with the domain R^2_+ x R^2_+ and its distinguished boundary R^2. Points in R^2_+ x R^2_+ are denoted by (y,t) where y ∈ R^2 and t = (t_1,t_2), t_1,t_2 > 0. The notation φ(u) is reserved for an even, real-valued, C_0(R) function with support in [-1,1] and such that

\[ \int_0^\infty \phi(u)^2 \frac{du}{u} = 1, \quad \left( \frac{du}{d\phi(0)} \right)^m = 0 \]

for sufficiently large m to be specified. Given such a φ, we let \( \Phi_i(y) = \phi(y_1/t_1)\phi(y_2/t_2)/t_1t_2 \). If f is a tempered distribution, we put \( f(y,t) = f \ast \Phi_i(y) \), and define the double square function \( Sf \) by

\[ Sf(x) = \left( \iint_{\Gamma(x)} |f(y,t)|^2 \frac{dt_1dt_2}{t_1^2t_2} \right)^{1/2} \]

where \( x = (x_1, x_2) \in R^2 \). Here \( \Gamma(x) \) denotes the product cone

\[ \Gamma(x) = \Gamma(x_1) \times \Gamma(x_2) = \{(y,t) : |x_1 - y_1| < t_1, |x_2 - y_2| < t_2 \} \].

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If \( m \sim (1/p - 1)_+ \), the Hardy space \( H^p(\mathbb{R}_+^2 \times \mathbb{R}_+^2) \), \( 0 < p < +\infty \), can be defined (modulo a normalization) as the set of tempered distributions \( f \) such that
\[
\| f \|_{H^p} = \| Sf \|_{L^p(\mathbb{R}^2)} < +\infty.
\]

Now let \( R_{y,t} \) denote the rectangle centered at \( y \) with dimensions \( t_1 \times t_2 \). With each open set \( \Omega \subset \mathbb{R}^2 \) we associate the Carleson region \( C(\Omega) = \{ (y, t): R_{y,t} \subset \Omega \} \). The space \( \text{BMO}(\mathbb{R}_+^2 \times \mathbb{R}_+^2) \) of functions of bounded mean oscillation is the set of all tempered distributions \( f \) (modulo a suitable class \( \mathcal{A} \) of distributions with Fourier-transforms supported on the coordinates axes) such that
\[
\| f \|_{\text{BMO}} = \sup_{\Omega} \left( \frac{1}{|\Omega|} \int_{C(\Omega)} |f(y, t)|^2 \frac{dy \, dt_1 \, dt_2}{t_1 t_2} \right)^{1/2} < +\infty.
\]

According to Chang-Fefferman [C-F], \( \text{BMO}(\mathbb{R}_+^2 \times \mathbb{R}_+^2) \) can be identified with the dual of \( H^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2) \). Moreover,
\[
(1.1) \quad \| f \|_{(H^p)^*} \approx \| f \|_{\text{BMO}}.
\]

Finally, we write down the definition of Peetre’s \( K \)-functional for the couple \( (H^p, \text{BMO}) \), \( 0 < p < +\infty \):
\[
K(t, f; H^p, \text{BMO}) = \inf_{f = f_0 + f_1} \max(\| f_0 \|_{H^p}, t \| f_1 \|_{\text{BMO}})
\]
defined for \( f \in H^p + \text{BMO} \) and \( t > 0 \).

2. Some auxiliary facts. A countable family \( \mathcal{O} = \{ \Omega_i \} \) of open sets \( \Omega_i \subset \mathbb{R}^2 \) of finite measure is called directed if either \( |\Omega_i \cap \Omega_j| = 0 \), \( \Omega_i \subset \Omega_j \) or \( \Omega_j \subset \Omega_i \) whenever \( \Omega_i, \Omega_j \in \mathcal{O} \). With such a family \( \mathcal{O} \) we associate the maximal operator \( M_\mathcal{O} \) defined by
\[
M_\mathcal{O} f(x) = \sup_{x \in \Omega} \frac{1}{|\Omega|} \int_{\Omega} |f(y)| \, dy, \quad \Omega_i \in \mathcal{O},
\]
whenever \( x \in \bigcup \Omega_i \), \( M_\mathcal{O} f(x) = 0 \) otherwise. In a standard fashion, it follows that \( M_\mathcal{O} \) is of weak-type \((1,1)\) and bounded on \( L^p \), \( 1 < p \leq +\infty \):
\[
(2.1) \quad t \{ M_\mathcal{O} f > t \} \lesssim \| f \|_{L^p},
\]
\[
(2.2) \quad \| M_\mathcal{O} f \|_{L^p} \lesssim c_p \| f \|_{L^p}, \quad 1 < p \leq +\infty.
\]

We also introduce the corresponding “local maximal operators” \( M_{\alpha, \mathcal{O}} \), \( 0 < \alpha < 1 \), defined by
\[
M_{\alpha, \mathcal{O}} f(x) = \sup_{x \in \Omega} \inf \{ A: \{ y \in \Omega: |f(y)| > A \} < \alpha |\Omega_i| \}, \quad \Omega_i \in \mathcal{O},
\]
cf. [J-T]. (In particular, when \( \alpha = \frac{1}{2} \), \( M_{\alpha, \mathcal{O}} f(x) \) is the supremum of the median-values of \( f \) over the sets \( \Omega_i \) containing \( x \).) Observing that \( \{ M_{\alpha, \mathcal{O}} f > t \} = \{ M_\mathcal{O} \chi_{|f| > t} \geq \alpha \} \), (2.1) gives us

**Lemma 2.1.** Let \( 0 < \alpha < 1 \). If \( \mathcal{O} \) is a directed family of open sets, then
\[
|\{ M_{\alpha, \mathcal{O}} f > t \}| \leq \left( \| f \|_{L^p} \right)^{1/\alpha}, \quad t > 0.
\]
Let $\Omega \subset \mathbb{R}^2$ be an open set and let $\Gamma_\Omega(x), x \in \mathbb{R}^2$, denote the truncated cone
\[ \Gamma_\Omega(x) = \{ (y, t) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : |x_i - y_i| < t_i, i = 1, 2, \text{ and } |R_{y,t} \cap \Omega| > \frac{1}{100} |R_{y,t}| \}. \]

Following Chang-Fefferman [C-F], we put
\[ S_\alpha f(x) = \left( \int \int_{\Gamma_\Omega(x)} |f(y, t)|^2 \frac{dy \, dt_1 \, dt_2}{t_1^2 t_2^2} \right)^{1/2}. \]

(Recall that $f(y, t)$ is a convolution, $f * \Phi_t(y)$; notice, however, that very little of the argument below uses this fact.) If $\mathcal{O} = \{ \Omega_i \}$ is a directed family of open sets, the "local square functions" $S_{\alpha, \mathcal{O}}, 0 < \alpha < 1,$ are defined by
\[ S_{\alpha, \mathcal{O}} f(x) = \sup \inf \left\{ A : \left| \left\{ y \in \Omega_i : S_\alpha \Omega_i f(y) > A \right\} \right| < \alpha |\Omega_i| \right\}, \quad \Omega_i \in \mathcal{O}. \]

The particular value of $\alpha$ is not important; the main reason for introducing it is that it makes it easier to express the subadditivity property
\[ S_{\alpha, \mathcal{O}} (f + g)(x) \leq 2 \left( S_{\alpha/2, \mathcal{O}} f(x) + S_{\alpha/2, \mathcal{O}} g(x) \right). \]

Clearly, $S_{\alpha, \mathcal{O}} f \leq M_{\alpha, \mathcal{O}}(f)$. Hence, by Lemma 2.1 and the definition of $H^p$,
\[ \|S_{\alpha, \mathcal{O}} f\|_{L^p} \leq c_{\alpha} \|f\|_{H^p}, \quad 0 < p < +\infty. \]

On the other hand, by Chebyshev's inequality,
\[ \left| \left\{ y \in \Omega_i : S_\alpha \Omega_i f(y) > A \right\} \right| \leq \int_{\Omega_i} S_\alpha \Omega_i f(y)^2 \frac{dy}{A^2} \]
\[ \leq c \int \int_{C(\hat{\Omega}_i)} |f(z, t)|^2 \frac{dz \, dt_1 \, dt_2}{t_1 t_2^2} / A^2, \]
where $\hat{\Omega}_i = \{ M \chi_{\Omega_i} > \frac{1}{100} \}$ and $M$ is the strong maximal operator. If $f \in \text{BMO}$, this is less than
\[ c \|f\|^2_{\text{BMO}} |\Omega_i| / A^2 \leq c \|f\|^2_{\text{BMO}} |\Omega_i| / A^2 \]
by the strong maximal theorem. Hence, when $A^2 > c \|f\|^2_{\text{BMO}} / \alpha$ we have $\left| \left\{ y \in \Omega_i : S_\alpha \Omega_i f(y) > A \right\} \right| < \alpha |\Omega_i|$, and consequently,
\[ \|S_{\alpha, \mathcal{O}} f\|_{L^\infty} \leq c_{\alpha} \|f\|_{\text{BMO}}. \]

Now select, once and for all, a family $\{O_k\}_{k \in \mathbb{Z}}$ of open sets $O_k \subset \mathbb{R}^2$ of finite measure with the following properties:
\[ \bigcup_k O_k = \mathbb{R}^2, \quad \bigcap_k O_k = \{0\}, \quad \{ M \chi_{O_{k+1}} > \frac{1}{4} \} \subset O_k, \quad k \in \mathbb{Z}. \]

Let $\Omega_k = O_k \setminus \overline{O}_{k+1}$ and let
\[ \mathcal{R}_k = \{ \text{all dyadic rectangles } R \text{ such that } |R \cap O_k| > \frac{1}{2} |R|, |R \cap O_{k+1}| \leq \frac{1}{2} |R| \}, \quad k \in \mathbb{Z}. \]

Observe that each dyadic rectangle belongs to exactly one $\mathcal{R}_k$. 

We will need the following simple lemma.

**Lemma 2.2.** If \( R \in \mathcal{R}_k \), then \( |R \cap \Omega_k| \geq \frac{1}{4}|R| \).

**Proof.** There are two possibilities: either \( |R \cap O_{k+1}| > \frac{1}{4}|R| \) or \( |R \cap O_{k+1}| \leq \frac{1}{4}|R| \). In the first case, \( R \subset \{ M \chi_{O_{k+1}} > \frac{1}{4} \} \subset O_k \) by (2.6). Hence,
\[
|R \cap \Omega_k| \geq |R \cap O_k| - |R \cap O_{k+1}| \geq |R| - \frac{1}{4}|R| = \frac{3}{4}|R|
\]
since \( R \in \mathcal{R}_k \). Similarly, in the second case,
\[
|R \cap \Omega_k| \geq |R \cap O_k| - |R \cap O_{k+1}| \geq \frac{1}{4}|R| - \frac{1}{4}|R| = \frac{1}{4}|R|
\]
This proves the lemma.

3. **The main result.** We are now ready to state our main result.

**Theorem 3.1.** Let \( 0 < p < +\infty \) and \( 0 < \alpha < 1 \). For each directed family \( \mathcal{O} \) of open sets
\[
K(t, S\alpha, \mathcal{O}; L^p, L^\infty) \leq c_\alpha K(t, f; H^p, \text{BMO}), \quad t > 0.
\]
Conversely, for each \( f \) there is a directed family \( \mathcal{O} = \mathcal{O}_f \) such that
\[
K(t, f; H^p, \text{BMO}) \leq c_{\alpha, p} K(t, S\alpha, \mathcal{O}; L^p, L^\infty), \quad t > 0.
\]

**Proof.** The first part follows immediately from (2.3)–(2.5).

Now recall that \( K(t, f; L^p, L^\infty)/t \) is the inverse of the best approximation functional
\[
E(t, f; L^p, L^\infty)/t = \inf_{\|f_1\|_{L^\infty} < t} \|f - f_1\|_{L^p}/t,
\]
and similarly for \( H^p \) and \( \text{BMO} \) (see [B-L, J-T]). Furthermore, it is easy to see that
\[
E(t, f; L^p, L^\infty) = \left( \int_{|f| > t} |f|^p \right)^{1/p}.
\]
Hence, to prove the second converse part it is enough to show that there is a family \( \mathcal{O} = \mathcal{O}_f \) such that

\[
E(t, f; H^p, \text{BMO}) \leq c \left( \int_{S\alpha, \mathcal{O} > t/c} |S\alpha, \mathcal{O}|^p \right)^{1/p}
\]
for some constant \( c \). By reiteration, it will be enough to do this for \( 0 < p < 1 \).

Let us first consider the construction of the directed family \( \mathcal{O} \). Let \( \{O_k\} \) be the family satisfying (2.6) that we selected earlier. \( \mathcal{O} = \{\Omega_{k,l}\}, \ k \in Z, \ l \in N, \) will be derived from \( \{O_k\} \) by an inductive argument:

We start by fixing \( k \in Z \). To get the induction started we define \( \Omega_{k,0} = \Omega_k = O_k \setminus \Omega_{k+1} \). Suppose \( \Omega_{k,0}, \ldots, \Omega_{k,l-1} \) have been chosen and put
\[
A_{k,l-1} = \inf\{A : \{x \in \Omega_{k,l-1} : S\alpha_{k,l-1} f(x) > A\} < \alpha|\Omega_{k,l-1}|\}.
\]
Then
\[
\Omega_{k,l} = \{x \in \Omega_{k,l-1} : S\alpha_{k,l-1} f(x) > A_{k,l-1}\}.
\]

We repeat this process for each \( k \). The sets \( \Omega_{k,l} \) obtained in this way satisfy \( \Omega_{k,l} \subset \Omega_{k,l-1} \), and since the sets \( \Omega_k = \Omega_{k,0} \) are pairwise disjoint, \( \mathcal{O} = \{\Omega_{k,l}\} \) is a directed family. Also, \( S\alpha_{k,l} f \in L^p + L^\infty \) implies that
\[
|\Omega_{k,l}| \leq \alpha|\Omega_{k,l-1}|.
\]
For \( k \in \mathbb{Z}, l \in \mathbb{N} \) we now define
\[
\mathcal{R}_{k,l} = \left\{ R \in \mathcal{R}_k : |R \cap \Omega_{k,l}| > \frac{1}{10} |R|, |R \cap \Omega_{k,l+1}| \leq \frac{1}{10} |R| \right\}.
\]
By Lemma 2.2 it follows that in this way each dyadic rectangle \( R \) belongs to a unique \( \mathcal{R}_{k,l} \). If \( R = I \times J \), we put \( R^+ = \{(y, t) \in \mathbb{R}^2 : y \in R, 2|I| < t_1 < 4|I|, 2|J| < t_2 < 4|J| \} \) and \( B^+_k = \bigcup R \in \mathcal{R}_k R^+ \).

The almost optimal decomposition of \( f = f_0 + f_1 \) required to prove (3.1) is given by
\[
f_0 = \sum_{k} \sum_{l \in L(k)} \int \int f(y, t) \Phi_l(x - y) \, dy \frac{dt_1 dt_2}{t_1 t_2} = \sum_{k} \sum_{l \in L(k)} b_{kl},
\]
where \( L(k) = L(k, t) = \{ l : \Omega_{k,l} \subset \{ S_{a,ef} > t \} \} \), and
\[
f_1 = \sum_{k} \sum_{l \in L(k)} \int \int f(y, t) \Phi_l(x - y) \, dy \frac{dt_1 dt_2}{t_1 t_2} = \sum_{k} \sum_{l \in L(k)} b_{kl}
\]
(with convergence in \( \mathcal{S}'/\alpha \)). Notice that, for each \( k \), \( L(k) \) is either empty or there is a unique \( l(k) \) such that \( \Omega_{k,l} \subset \{ S_{a,ef} > t \} \) if \( l \geq l(k) \).

By checking Fourier transforms it follows that \( f = f_0 + f_1 \) (in \( \mathcal{S}'/\alpha \)). Hence, to prove (3.1) we need to verify that
\[
\tag{3.3}
\| f_0 \|^p_{\mathcal{H}^p} \leq c \int_{S_{a,ef} > t} |S_{a,ef}|^p
\]
and
\[
\tag{3.4}
\| f_1 \|_{BMO} \leq ct.
\]
We claim that for each \( k \) and \( l \) \( b_{kl} \) is a \( p \)-atom (in the sense of [C-F]) after the appropriate normalization \( cb_{kl}(x)/\sup_{x \in \Omega_{k,l+1}} S_{a,ef}(x) |\Omega_{k,l}|^{1/p} \). This would imply that
\[
\| f_0 \|^p_{\mathcal{H}^p} \leq \sum_{k} \sum_{l \in L(k)} \| b_{kl} \|^p_{\mathcal{H}^p}
\]
\[
\leq c \sum_{k} \sum_{l \in L(k)} \left( \sup_{x \in \Omega_{k,l}} S_{a,ef}(x) \right)^p |\Omega_{k,l}|.
\]
By (3.2),
\[
|\Omega_{k,l}| \leq c |\Omega_{k,l} \setminus \Omega_{k,l+1}| = c \left| \Omega_{k,l} \setminus \bigcup_{m > l} \Omega_{k,m} \right|
\]
since \( \Omega_{k,m} \subset \Omega_{k,l+1} \) if \( m > l \). Also notice that \( S_{a,ef}(x) \) is an elementary function which is constant on \( \Omega_{k,l} \setminus \Omega_{k,l+1} \). Hence,
\[
\| f_0 \|^p_{\mathcal{H}^p} \leq c \sum_{k} \sum_{l \in L(k)} \int_{\Omega_{k,l} \setminus \bigcup_{m > l} \Omega_{k,m}} S_{a,ef}(x)^p \, dx
\]
\[
\leq c \sum_{k} \int_{\{ S_{a,ef} > t \} \cap \Omega_{k}} S_{a,ef}(x)^p \, dx
\]
\[
= c \int_{\{ S_{a,ef} > t \}} S_{a,ef}(x)^p \, dx,
\]
and this would prove (3.3).
The argument needed to verify our claim is essentially the same as that in [C-F2, J-T2] to which we refer for more details.

The main step is to show that

\[(3.5) \quad \| b_{kl} \|_{L^2} \leq c \sup_{x \in \Omega_{kl} \setminus \Omega_{kl+1}} S_{\alpha, \epsilon f}(x) |\Omega_{kl}|^{1/2}.
\]

To see this we pick \( g \in L^2(\mathbb{R}^2) \) and put

\[\omega_m = \omega_{klm} = \left\{ M_{X_{\Omega_{kl}}} > \frac{1}{100} \right\} \cap \Omega_{k,m} \setminus \Omega_{k,m+1}.\]

By Cauchy-Schwarz and the fact that

\[\mathbb{E} |H_{\mu} : x, y, t| \leq c\]

when \((y, t) \in B_{k, l}^+\), we have

\[
\int b_{kl}(x) g(x) \, dx = \int \int_{B_{k, l}^+} b_{kl}(y, t) g(y, t) \, dy \, dt \leq c \sum_{m \leq l} \int \omega_{m} f(x) S_{\Omega_{kl}} g(x) \, dx.
\]

The way in which the sets \( \Omega_{kl} \) were defined implies that this is less than

\[
c \int_{\mathbb{R}^n} \left( \sum_{m \leq l} S_{\alpha, \epsilon f}(x) S g(x) \right) \, dx \leq c \sup_{x \in \Omega_{kl} \setminus \Omega_{kl+1}} \left( \sum_{m \leq l} \omega_{m} \right)^{1/2} \| S g \|_{L^2}
\]

\[
\leq c \sup_{x \in \Omega_{kl} \setminus \Omega_{kl+1}} \left( \sum_{m \leq l} \omega_{m} \right)^{1/2} \| g \|_{L^2}.
\]

Now (3.5) readily follows since \( \sum_{m \leq l} \omega_{m} \subset \left\{ M_{X_{\Omega_{kl}}} > \frac{1}{100} \right\} \cap \Omega_k \setminus \Omega_{k,l+1} \) and, hence, by the strong maximal theorem \( |\sum_{m \leq l} \omega_{m}|^{1/2} \leq c |\Omega_{kl}|^{1/2} \).

There remains to prove (3.4). Proceeding as in the proof of (3.5) we find that

\[(3.6) \quad \langle g, f_i \rangle \leq c \int_{S_{\alpha, \epsilon f}} S_{\alpha, \epsilon f}(x) S g(x) \, dx \leq c \| g \|_{H^1},
\]

for each Schwartz function \( g \) such that the support of \( \dot{g} \) has a positive distance to the coordinate axes. This means that \( f_i \) defines a linear functional on \( H^1 \) and, by (1.1), \( \| f_i \|_{BMO} \leq c \). This completes the proof of (3.4) and the theorem.

Suppose \( f \in \mathcal{S}' \backslash \mathcal{A} \) and let \( \partial = \partial f \) be the directed family of open sets constructed in the proof above.

**Corollary 3.2.** Let \( 0 < \alpha < 1 \). Then

\[(3.7) \quad \| f \|_{H^p} \approx \| S_{\alpha, \epsilon f} \|_{L^p}, \quad 0 < p < +\infty,
\]

and

\[(3.8) \quad \| f \|_{BMO} \approx \| S_{\alpha, \epsilon f} \|_{L^\infty}.
\]

**Proof.** By referring to (2.4) and (2.5) we take care of one way. To show the converse inequalities, we first consider (3.7) in the case \( 0 < p < 1 \): The proof of (3.8) shows that if \( S_{\alpha, \epsilon f} \in L^p \), then \( f \) has an atomic decomposition into \( p \)-atoms. Hence, \( \| f \|_{H^p} \leq c \| S_{\alpha, \epsilon f} \|_{L^p} \). Formally this corresponds to \( i = 0 \) in (3.3).
The proof of (3.6) on the other hand shows that
\[
(g, f) \leq c \int S_a \circ f(x) S_g(x) \, dx \leq c \|S_a \circ f\|_{L^p} \|g\|_{L^p},
1 < p < +\infty.\] This readily gives us (3.7) for $1 < p < +\infty$.

The inequality $\|f\|_{BMO} \leq c \|S_a \circ f\|_{L^\infty}$ follows in a similar way. At least formally it is obtained by replacing $f_1$ by $f$ and by putting $t = \infty$ in (3.6).

Combining Theorem 3.1 and (3.7) we obtain the following characterization, due to K. C. Lin [L], of the real interpolation spaces between $H^p$ and BMO:

**Corollary 3.3.** Let $0 < p_0 < +\infty$ and $0 < \theta < 1$. Then $(H^{p_0}, BMO)_{\theta p} = H^p/\sigma^\theta$, $1/p = (1 - \theta)/p_0$, with equivalent (quasi-) norms.

**References**


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