LU-FACTORIZATION OF OPERATORS ON $l_1$

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ABSTRACT. Necessary and sufficient conditions are obtained for $LU$-factorization of operators on $l_1$. In particular it is shown that uniform invertibility of the compressions of the operator is not sufficient to insure an $LU$-factorization of the operator, thus answering a question of de Boor, Jia, and Pinkus.

The question of when a bounded linear operator on $l_p$, $1 \leq p \leq \infty$, has an $LU$-factorization has been much studied recently. Barkar and Gohberg [2] have shown that if $A$ is an operator on $l_p$ which has an $LU$-factorization, then $A$ and its compressions $A_n = P_n A P_n$ are uniformly invertible, i.e. $\sup_n (\|A_n^{-1}\|, \|A^{-1}\|) < \infty$. In the other direction, various classes of operators such as invertible, diagonally dominant operators on $l_1$ [7] and invertible, totally positive operators [3, 1] on $l_p$ have been shown to have $LU$-factorizations. For these kinds of operators it is known [1] that their compressions satisfy a stronger condition than uniform invertibility; namely, that the inverses of the compressions are order bounded, i.e. $\|\sup_n A_n^{-1}\| < \infty$. Left open, then, is the possibility (first raised in [3] with a negative expectation) that uniform invertibility might be sufficient for a matrix operator on $l_\infty$ to have an $LU$-factorization. In this paper an example is given that shows that uniform invertibility is not sufficient for factoring an operator on $l_\infty$ (or $l_1$). However, we also show that uniform invertibility of the compressions is sufficient to ensure an $LU$-factorization when the operator has an inverse whose columns decay at a certain rate away from the diagonal. Among the operators with this property are the banded operators.

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We now fix some terminology and notation. If $x = (x_i)$ is an element of $l_1$ we denote its usual projection onto the span of the first $n$ basis vectors by $P_n x$. A bounded linear operator $A$ on $l_1$ is said to be upper (respectively lower) triangular if $P_n A P_n = A P_n$ (respectively $P_n A$) for all $n$. We say that $A$ is unit upper (lower) triangular if it is upper (lower) triangular and its diagonal entries in the matrix representation for $A$ relative to the usual basis $e_i$ of $l_1$ are all ones. An operator $A$ is said to have an $LU$-factorization (relative to the usual basis $e_i$ of $l_1$) if there exist invertible operators $L$ and $U$ so that $A = LU$ and the operators $L, L^{-1}$ are unit lower triangular while $U, U^{-1}$ are upper triangular. An operator $A$ is said to be

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banded if there exist integers \( m \) and \( l \) so that \( A(i, j) = 0 \) if \( j \not\in [i - l, i - l + m] \).

The absolute value of an operator \( A = (a_{ij}) \) is the operator \( |A| = (|a_{ij}|) \). Finally, we let \( A_n^{-1} \) denote the operator on \( l_1 \) whose decomposition with respect to \( P_n \) and \( I - P_n \) is given by

\[
\begin{pmatrix}
(P_n A P_n)^{-1} & 0 \\
0 & 0
\end{pmatrix}.
\]

**Example.** For each \( m \), let \( B_m \) be the operator on \( l_1^m \) given by \( B_m e_j = e_1 - e_{j+1} \), \( j = 1, 2, \ldots, m - 1 \), and \( B_m e_m = e_1 \). Then each \( B_m \) is invertible relative to \( l_1^m \); in fact, \( B_m^{-1} e_1 = e_m \) and \( B_m^{-1} e_j = e_m - e_{j-1} \), \( j = 2, 3, \ldots, m \). Since for each \( i \), \( P_i B_m P_i = B_i \), we have that the compressions of each \( B_m \) are invertible and so each \( B_m \) has an \( LU \)-factorization [4, p. 178]. In fact, \( B_m = L_m U_m \) where \( L_m e_j = e_j - e_{j+1}, j = 1, 2, \ldots, m - 1 \), and \( L_m e_m = e_m \) and \( U_m e_j = \sum_{k=1}^j e_k \), \( j = 1, 2, \ldots, m \). Note that \( \|U_m\| = m \). If we now let \( A = \oplus_{m=1}^\infty B_m \) then \( A \) and its compressions are uniformly invertible; in fact, \( \sup_n (\|A_n^{-1}\|, \|A_n^{-1}\|, \|A_n\|) = 2 \). But if \( A = LU \) then \( \|U\| \geq \sup_n \|P_n U P_n\| \geq \sup_n \|U_m\| = \infty \), so \( A \) does not have an \( LU \)-factorization. This fact can also be easily obtained using Theorem 2 of [1] since \( B_m^{-1} e_1 = e_m \) implies that \( (\sup_m B_m^{-1}) e_1 = \sum_m e_m \), i.e. \( \sup_m B_m^{-1} = \infty \). Consequently, the block diagonal matrix \( A \) must also have \( \|\sup_n A_n^{-1}\| \) and so does not have an \( LU \)-factorization. We remark that \( A^*: l_\infty \to l_\infty \) does not have an \( LU \)-factorization either. For if \( A^* = LU \), since \( L \) and \( U \) are operators on \( l_\infty \), representable as matrices, \( A = U_* L_* \) is an \( LU \)-factorization for \( A \) where \( U_* \) and \( L_* \) are the preadjoints of \( U \) and \( L \) [8]. This fulfills the expectation raised in [3].

The question remains as to whether there are any easily recognized situations in which uniform invertibility of the compressions is sufficient to insure an \( LU \)-factorization of the operator. In order to give an example of such a situation we find it convenient to give a characterization of when an operator on \( l_1 \) has an \( LU \)-factorization. This characterization is similar to that presented in Theorem 2 of [1] where the finiteness of \( \|\sum |A_n^{-1} - A_n'\| \) is replaced by the finiteness of \( \|\sup_n A_n^{-1}\| \). As further motivation we recall that if an operator \( A \) and its compressions are uniformly invertible, then \( A_n^{-1} e_i \to A^{-1} e_i \) for all \( i \). Our first result shows that for \( A \) to have an \( LU \)-factorization this convergence must be of a telescoping variety.

**Theorem 1.** A bounded linear operator \( A \) on \( l_1 \) has an \( LU \)-factorization if and only if, for each \( n \), \( A_n = P_n A P_n \) is invertible and

\[
\sup_i \sum_{n=1}^\infty \| (A_n^{-1} - A_n^{-1}) e_i \| = \left\| \sum_{n=1}^\infty |A_n^{-1} - A_n^{-1}| \right\| < \infty.
\]

**Proof.** If \( A = LU \) then \( A_n = P_n L P_n U P_n \) and hence \( A_n^{-1} = P_n U^{-1} P_n L^{-1} P_n = U^{-1} P_n L^{-1} \) since \( U^{-1} \) is upper triangular and \( L^{-1} \) is lower triangular. Consequently, \( (A_n^{-1} - A_n^{-1})(e_i) = U^{-1}(P_n^{-1} - P_n)L^{-1} e_i \), so

\[
\sup_i \sum_{n=1}^\infty \| (A_n^{-1} - A_n^{-1})(e_i) \| \leq \sup_i \| U^{-1} \| \sum_{n=1}^\infty \| (P_n^{-1} - P_n) L^{-1} e_i \|
\]

\[
\leq \| U^{-1} \| \sup_i \| L^{-1} e_i \| = \| U^{-1} \| \| L^{-1} \| < \infty.
\]
For the converse, note that the hypothesis implies that

\[ B e_i = A_1^{-1} e_i + \sum_{n=1}^{\infty} (A_{n+1}^{-1} - A_n^{-1}) e_i \]

exists for each \( i \) and \( \sup\|B e_i\| < \infty \). Hence \( B \) extends to a bounded linear operator on \( l_1 \) and since \( B e_i = \lim_n A_n^{-1} e_i \) it follows quickly that \( B = A^{-1} \). Now for each \( N \),

\[ A^{-1} e_i = A_N^{-1} e_i + \sum_{n=N}^{\infty} (A_{n+1}^{-1} - A_n^{-1}) (e_i) \]

and so

\[ A_N^{-1} = A^{-1} + \sum_{n=N}^{\infty} (A_{n+1}^{-1} - A_n^{-1}) \]

pointwise. Hence

\[ \sup_N |A_N^{-1}| \leq |A^{-1}| + \sum_n |A_{n+1}^{-1} - A_n^{-1}| \]

pointwise and, consequently,

\[ \left\| \sup_N |A_N^{-1}| \right\| \leq \|A^{-1}\| + \left\| \sum_n |A_{n+1}^{-1} - A_n^{-1}| \right\| < \infty. \]

Now since \( A_n \) is invertible for all \( n \), we have that \( A_n = L_n U_n \). We shall show that the operators \( L_n^{-1} \) and \( U_n^{-1} \) are bounded and so deduce that \( A \) has an \( LU \)-factorization. (This part of the argument has already appeared in [1] but we include it here for the sake of completeness.) Now for each \( n \),

\[ L_n^{-1}(i, j) = -\sum_{k=1}^{i-1} A_{i-1}^{-1}(k, j) A(i, k) \quad \text{for } i > j \]

and

\[ U_n^{-1}(i, j) = A_j^{-1}(i, j) \quad \text{for } i < j \]

[1, 2]. It follows that

\[ \sup_n |L_n^{-1}(i, j)| \leq \sum_{k=1}^{\infty} \sup_i |A_{i-1}^{-1}(k, j)| |A(i, k)| \quad \text{for } i > j \]

and so

\[ \sup_n \|L_n^{-1}\| \leq \left\| \sup_n |L_n^{-1}| \right\| \leq \sup_i \|A_{i-1}^{-1}\| \|A\| + 1 < \infty. \]

Similarly,

\[ \sup_i \|U_n^{-1}\| \leq \left\| \sup_n |U_n^{-1}| \right\| \leq \sup_n \|A_n^{-1}\| \leq \infty. \]

Since \( L_n = P_n L_{n+1} P_n \) and \( U_n = P_n U_{n+1} P_n \) we have that \( L_n^{-1} = P_n L_{n+1}^{-1} P_n \) and \( U_n^{-1} = P_n U_{n+1}^{-1} P_n \). Consequently, for each \( x \) in \( l_1 \), the limits \( \lim_n L_n x = Lx \), \( \lim_n U_n^{-1} x = Vx \), \( \lim_n U_n x = Ux \), and \( \lim_n U_n^{-1} x = Wx \) exist and define bounded triangular operators on \( l_1 \). Now since

\[ LVx = \lim_n L_n L_n^{-1} x = \lim L_n x = x = \lim I_n x = \lim L_n^{-1} L_n x = VLx \]
we have that $V = L^{-1}$. Similarly, $W = U^{-1}$. Finally, for each $x$ in $l_1$, we have that $LUx = \lim_n L_n U_n x = \lim_n A_n x = Ax$ so $A$ has the promised factorization.

We remark that Theorem 1 can be easily applied to the example preceding the theorem. In this case

$$
A_2^{-1} - A_1^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3^{-1} - A_2^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix},
$$

$$
A_4^{-1} - A_3^{-1} = \begin{pmatrix} 0 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad A_5^{-1} - A_4^{-1} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix},
$$

$$
A_6^{-1} - A_5^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.
$$

(Here we have displayed only the upper left hand, nonzero portion of each operator.) Hence

$$
\sum_{n=1}^{\infty} \| (A_{n+1}^{-1} - A_n^{-1})(e_2) \| = 3, \quad \sum_{n=1}^{\infty} \| (A_{n+1}^{-1} - A_n^{-1})(e_4) \| = 5
$$

and, in general, by a routine but tedious induction argument,

$$
\sum_{n=1}^{\infty} \| (A_{n+1}^{-1} - A_n^{-1})(e_{k(k+1)/2+1}) \| = 2k + 1
$$

so by Theorem 1 $A$ does not have an $LU$-factorization.

**Theorem 2.** Let $A$ be a bounded linear operator on $l_1$. If $A$ and its compressions are uniformly invertible and, in addition,

$$
sup_n \max_{k \leq n} \| A^{-1}(i, i+k, i) \| < \infty,
$$

then $A$ has an $LU$-factorization.

**Proof.** We start with $AA^{-1}e_i = e_i$. Hence

$$
AP_n A^{-1}e_i + A(I - P_n)A^{-1}e_i = e_i \quad \text{for all } n
$$

so

$$
P_n AP_n A^{-1}e_i + P_n (I - P_n) A^{-1}e_i = P_n e_i = e_i \quad \text{for } i \leq n.
$$

Hence

$$
P_n A^{-1}e_i - A_n^{-1}e_i = A_n^{-1}P_n A(I - P_n) A^{-1}e_i \quad \text{for } i \leq n
$$

and thus

$$
\| P_n A^{-1}e_i - A_n^{-1}e_i \| \leq M^2 \| (P_n - I) A^{-1}e_i \| \quad \text{for } i \leq n
$$

where $M = \sup_n \{ \| A_n^{-1} \|, \| A \| \}$. Now

$$
\| A_n^{-1}e_i - A^{-1}e_i \| \leq \| P_n A^{-1}e_i - A_n^{-1}e_i \| + \| (I - P_n) A^{-1}e_i \|
$$

$$
\leq (M^2 + 1) \| (I - P_n) A^{-1}e_i \| \quad \text{for } i \leq n.
$$
Hence
\[
\sum_{n=1}^{\infty} \| A_{n+1}^{-1} e_i - A_n^{-1} e_i \| \leq 2(M^2 + 1) \sum_{n=1}^{\infty} \| (I - P_n) A_n^{-1} e_i \|
\]
\[
\leq 2(M^2 + 1) \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} | \langle A_n^{-1} e_i, e_j \rangle |
\]
\[
\leq 2(M^2 + 1) \sum_{k=1}^{\infty} k | \langle A_{k+1}^{-1} e_i, e_{i+k} \rangle |.
\]

Consequently,
\[
\sup_{i} \sum_{n=1}^{\infty} \| (A_{n+1}^{-1} - A_n^{-1}) e_i \| \leq \sup_i \| A_i^{-1} e_i \| + \sup_i \sum_{n=1}^{\infty} \| (A_{n+1}^{-1} - A_n^{-1})(e_i) \| < \infty.
\]

So by Theorem 1 A has an LU-factorization.

It is not surprising that the additional condition imposed in Theorem 2 is far from necessary. For example, choose a so that 0 < a < 1 and \( \sum_n (a/n^2) < 1 \) and let
\[
B(i,j) = \begin{cases} a/(i-1)^2, & j = 1, i \neq 1, \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( \| B \| < 1 \), and \( B(i,i) = 0 \) for all i so \( A = I - B \) is invertible and strictly (column) diagonally dominant. Consequently, A has an LU-factorization [7]. But since \( B^2 = 0 \), \( A^{-1} = I + B \) and so
\[
\sum_{k=1}^{\infty} k | A^{-1}(k + 1, 1) | = \sum_{k=1}^{\infty} k \frac{a}{k^2} = \infty.
\]

The problem here is the slowness of the decay rate of the entries of \( A^{-1} \) away from the main diagonal; however, for banded operators this poses no difficulty.

**Corollary 3.** Let A be a banded operator on \( l_1 \). Then A has an LU-factorization if and only if A and its compressions are uniformly invertible.

**Proof.** One direction is clear; for the other we recall from [5] that if A is banded and invertible then there are positive constants C and \( \lambda \) with \( \lambda < 1 \) so that \( |A^{-1}(i,j)| \leq C \lambda^{j-i} \) for all \( i, j \). Consequently,
\[
\sup_i \sum_{k=1}^{\infty} k | A^{-1}(i + k, i) | \leq C \sum_{k=1}^{\infty} k \lambda^k < \infty.
\]

So Theorem 2 shows that A has an LU-factorization.

**References**


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