STRUCTURAL THEOREMS FOR PERIODIC ULTRADISTRIBUTIONS

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Abstract. Structural theorems for periodic ultradistributions of Roumieu and Beurling types are given.

1. Introduction. Periodic ultradistributions spaces $\mathcal{D}'(m_k)$ and $\mathcal{D}'(m_k)$ of Roumieu type and Beurling type, respectively, are investigated in [1 and 2]. In the mentioned papers periodic ultradistributions are characterized by the growth rate of their coefficients. We notice that these spaces are subspaces of the space $\mathcal{P}'$ of periodic generalized functions investigated in [5].

In this paper we shall give representation theorems for elements from $\mathcal{D}'(m_k)$ and $\mathcal{D}'(m_k)$.

2. Notations and preliminary results. In this part we shall recall basic notions from [2]. Also, we shall use notations from [2].

Sequence $m_k$ is a sequence of positive numbers for which in [2] it is supposed:
1. $m_k \leq m_{k+1}^2$, $k = 1, 2, \ldots$;
2. $\sum_{k=1}^{\infty} m_{k-1}/m_k < \infty$;
3. There are constants $A$ and $H$ such that $m_{k+1} \leq AH^k m_k$, $k = 0, 1, \ldots$.
4. There are constants $A$ and $H$ such that

\[ m_k \leq AH^k \min_{0 \leq q < k} m_q m_{k-q}, \quad k = 0, 1, \ldots. \]

We note that these conditions and some others are analyzed in [3] in detail. If $s > 1$, the Gevrey sequence $m_k = (k!)^s$ or $k^{ks}$ or $\Gamma(1+ks)$ satisfies the above conditions.

Condition (1) directly implies

\[ m_q m_k \leq m_0 m_{k+q}, \quad q = 0, 1, \ldots, k = 0, 1, \ldots. \]

Thus condition (4) from [2] is superfluous.

Conditions (1)–(4) imply “good” properties of periodic ultradistributions spaces (see [2]). For our investigations we shall assume that the sequence $m_k$ satisfies conditions (1), (3) and instead of (2) we shall assume

\[ (2^*) \lim_{k \to \infty} \frac{1}{m_k} = \infty. \]
Since \( m_k = m_0(m_1/m_0) \cdots (m_k/m_{k-1}) \), (2) implies (2)*. If \( m_k = (k!)^\alpha \), \( 0 < \alpha \leq 1 \), then (2)* holds but (2) does not.

One can easily prove that (2)* is equivalent to the fact that the space \( \mathcal{D} \) of polynomials of the form \( \sum_{k=0}^{\infty} c_k e_k(t) \), \( c_k \) are complex numbers and \( e_k = \exp(ikt) \), \( k = 0, \pm 1, \ldots \), is a subspace of \( \mathcal{D}(m_k) \) and therefore of \( \mathcal{D}\{m_k\} \) (see below).

As in [2] we denote by \( \mathcal{D}(m_k, L) \), \( L > 0 \), the space of all smooth functions on the unit circle \( K \), such that \( \phi \in \mathcal{D}(m_k, L) \) iff

\[
\left\| \phi \right\|_{L, \infty} := \sup \left\{ \left\| \phi^{(k)} \right\|_\infty / L^k m_k ; k = 0, 1, \ldots \right\} < \infty
\]

\[
\left\| \phi \right\|_\infty = \sup \{|\phi(t)| ; t \in [0, 2\pi]\}
\]

The space \( \mathcal{D}(m_k, L) \) is a \((B)\)-space. Spaces \( \mathcal{D}(m_k) \) and \( \mathcal{D}(m_k) \) are defined as

\[
\mathcal{D}\{m_k\} = \text{ind lim}_{L \to \infty} \mathcal{D}(m_k, L), \quad \mathcal{D}(m_k) = \text{proj lim}_{L \to 0} \mathcal{D}(m_k, L),
\]

where we take these limits in the topological sense. In the sense of strong topologies in \( \mathcal{D}'(m_k) \) and \( \mathcal{D}'(m_k) \) we have [2],

\[
\mathcal{D}'(m_k) = \text{proj lim}_{L \to \infty} \mathcal{D}'(m_k, L), \quad \mathcal{D}'(m_k) = \text{ind lim}_{L \to 0} \mathcal{D}'(m_k, L).
\]

In [6, Chapter 9], Zemanian investigated spaces \( \mathcal{A} \) and \( \mathcal{A}' \). As a particular case he investigated spaces \( \mathcal{A} \) and \( \mathcal{A}' \) which correspond to the space \( L^2(0, 2\pi) \) and the differential operator \( \mathcal{D} = iD \) (\( D \) is the derivative). Let us denote these spaces by \( \mathcal{A}_{\text{per}} \) and \( \mathcal{A}'_{\text{per}} \). We remark that these spaces may be identified with the spaces \( \mathcal{A} \) and \( \mathcal{A}' \) (for the definition of the spaces \( \mathcal{A} \) and \( \mathcal{A}' \) see [1]).

If \( \phi \) belongs to \( \mathcal{D}(m_k) \) or \( \mathcal{D}(m_k) \) its Fourier coefficients

\[
c_k(\phi) = \langle \phi, e_k(-t) \rangle = \int_0^{2\pi} \phi(t) e_k(-t) \, dt, \quad k = 0, \pm 1, \ldots,
\]

decrease more rapidly than any power of \(|k|\) when \(|k| \to \infty [2, \text{p. 146}]\). This implies that \( \mathcal{D}(m_k) \) and \( \mathcal{D}(m_k) \) are subspaces of \( \mathcal{A}_{\text{per}} \) (see [6, Theorem 9.3.3]). The space \( \mathcal{A}_{\text{per}} \) consists of elements from \( L^2(0, 2\pi) \cap C^\infty(0, 2\pi) \) for which

\[
\gamma_p(\phi) := \sup \{ \left\| D'^i \phi \right\|_2 ; i \leq p \} < \infty, \quad p = 1, 2, \ldots
\]

\[
\left\| \phi \right\|_2 \left( \int_0^{2\pi} |\phi(t)|^2 \, dt \right)^{1/2}
\]

and

\[
\langle D^k \phi, e_n(t) \rangle = (-in)^k \langle \phi, e_n(t) \rangle, \quad k = 0, 1, \ldots, \quad n = 0, \pm 1, \ldots.
\]

**Theorem 1.** \( \mathcal{D}(m_k) \) and \( \mathcal{D}(m_k) \) can be identified with \( \mathcal{A}_{\text{per}}(m_k) \) and \( \mathcal{A}'_{\text{per}}(m_k) \) respectively, where

\[
\mathcal{A}_{\text{per}}(m_k) = \text{ind lim}_{L \to \infty} \mathcal{A}_{\text{per}}(m_k, L), \quad \mathcal{A}'_{\text{per}}(m_k) = \text{proj lim}_{L \to 0} \mathcal{A}'_{\text{per}}(m_k, L),
\]

and

\[
\mathcal{A}_{\text{per}}(m_k, L) = \left\{ \phi \in \mathcal{A}_{\text{per}} ; \left\| \phi \right\|_{L, 2} := \sum_{k=0}^{\infty} \left\| \phi^{(k)} \right\|_2 / L^k m_k < \infty \right\}.
\]
Proof. If \( \phi \in \mathcal{D}(m_k, L/2) \), then \( \phi \in \mathcal{A}_{\text{per}}(m_k, L) \). Let \( \phi(t) = \sum_{k=-\infty}^{\infty} a_k e_k(t) \in \mathcal{A}_{\text{per}}(m_k, L) \). This series converges uniformly to \( \phi \) on \([0, 2\pi]\). The same holds for

\[
\phi'(t) = \sum_{k=-\infty}^{\infty} a_k (ik) e_k(t), \quad t \in [0, 2\pi].
\]

We have

\[
\|\phi\|_{\infty} \leq \sum_{k=-\infty}^{\infty} |a_k| \leq \left( \sum_{k=-\infty}^{\infty} a_k^2 k^2 \right)^{1/2} \left( \sum_{k=-\infty}^{\infty} k^{-2} \right)^{1/2} = C\|\phi'\|_2
\]

where \( C = 2\sum_{k=0}^{\infty} k^{-2} \). Using (3) we obtain

\[
\frac{\|\phi\|_{\infty}}{L^k m_k} \leq \frac{AC}{H} \left( \frac{L}{H} \right)^{k+1} m_{k+1}^{1/2}.
\]

Thus if \( \phi \in \mathcal{A}_{\text{per}}(m_k, L/H) \) then \( \phi \in \mathcal{D}(m_k, L) \). It is easy now to prove the assertion of Theorem 1.

3. Representation theorems.

Theorem 2. If \( f \in \mathcal{D}'(m_k) \) then there exist a sequence of functions \( f_i, i = 0, 1, \ldots \), on \((0, 2\pi)\) and a natural number \( n \) such that

(i) \( f_i \in L^2(0, 2\pi), i = 0, 1, \ldots \);

(ii) \( \sup_i \| f_i \|_2 < \infty \);

(iii) \( f(t) = \sum_{n=0}^{\infty} (n^i/m_i) f_i^{(i)}(t) \).

Conversely, if \( f_i, i = 0, 1, \ldots \), is a sequence of functions on \((0, 2\pi)\) for which (i) and (ii) hold, then by the series on the right side of (iii) a unique element from \( \mathcal{D}'(m_k) \) is defined.

The convergence of the series in (iii) is understood in a weak sense.

In the proof of Theorem 2 we shall show that \( \mathcal{D}(m_k) \) is a projective limit of a reduced compact sequence of \((\mathfrak{B})\)-spaces. Thus \( \mathcal{D}(m_k) \) and \( \mathcal{D}'(m_k) \) are Montel spaces and the weak and strong sequential convergence in \( \mathcal{D}'(m_k) \) are coincident.

Proof. We denote by \( \mathcal{A}_{\text{per}}(m_k, 1/n) \) the subspace of \( \mathcal{A}_{\text{per}} \) such that

\[
\phi \in \mathcal{A}_{\text{per}}(m_k, 1/n) \iff \|\phi\|_{1/n, 2} < \infty,
\]

where \( n \) is a fixed natural number.

Let \( \phi_p \) be a Cauchy sequence in \( \mathcal{A}_{\text{per}}(m_k, 1/n) \). Since \( \mathcal{A}_{\text{per}} \) is complete there is a \( \phi_0 \in \mathcal{A}_{\text{per}} \) such that (by Fatou’s Lemma)

\[
\sum_{k=0}^{\infty} \frac{n^k \|\phi_p^{(k)} - \phi_0^{(k)}\|_2}{m_k} = \sum_{k=0}^{\infty} \frac{n^k}{m_k} \lim_{\mu \to \infty} \|\phi_p^{(k)} - \phi_0^{(k)}\|_2 \leq \liminf_{\mu} \sum_{k=0}^{\infty} \frac{n^k}{m_k} \|\phi_p^{(k)} - \phi_0^{(k)}\|_2.
\]

This inequality implies that \( \mathcal{A}_{\text{per}}(m_k, 1/n) \) is complete. If \( \phi \in \mathcal{A}_{\text{per}}(m_k, 1/n) \) and \( \phi(t) = \sum_{k=-\infty}^{\infty} c_k e_k(t) \), then the sequence \( \sum_{n=0}^{\infty} c_k e_k(t) \), \( n = 0, 1, \ldots \), converges to \( \phi \) in the norm \( \| \cdot \|_{1/n, 2} \). Since for every \( n \) \( \sum_{k=-n}^{n} c_k e_k(t) \in \mathcal{A}_{\text{per}}(m_k) \) it follows that \( \mathcal{A}_{\text{per}}(m_k, 1/n) \) is the completion of \( \mathcal{A}_{\text{per}}(m_k) \) in the sense of the norm \( \| \cdot \|_{1/n, 2} \).
The sequence of norms \( \| \cdot \|_{1/n} \), \( n = 1, 2, \ldots \), is equivalent with the sequence of norms \( \| \cdot \|_{1/n, \infty} \) on \( \mathcal{A}_{\text{per}}(m_k) \). If we denote by \( \mathcal{A}_{\text{per}}(m_k, 1/n) \) the completion of \( \mathcal{A}_{\text{per}}(m_k) \) in the sense of the norm \( \| \cdot \|_{1/n, \infty} \), \( \mathcal{A}_{\text{per}}(m_k, 1/n) \) is identical with the space \( \mathcal{B}(m_k, 1/n) \).

From Theorem 1 it follows that for any \( n_0 \) there exist \( n_1, n_2, \) and \( n_3, n_0 < n_1 < n_2 < n_3 \), such that the inclusion mappings

\[
\mathcal{A}_{\text{per}}(m_k, 1/n_3) \rightarrow \mathcal{A}_{\text{per}}(m_k, 1/n_2) \rightarrow \mathcal{A}_{\text{per}}(m_k, 1/n_1) \rightarrow \mathcal{A}_{\text{per}}(m_k, 1/n_0)
\]

are continuous. By the Ascoli-Arcela theorem we obtain that the mapping \( \mathcal{A}_{\text{per}}(m_k, 1/n_2) \rightarrow \mathcal{A}_{\text{per}}(m_k, 1/n_1) \) is compact. It implies that the mapping \( \mathcal{A}_{\text{per}}(m_k, 1/n_3) \rightarrow \mathcal{A}_{\text{per}}(m_k, 1/n_0) \) is compact as well. Thus we obtain (see [3, p. 33])

\[
\mathcal{B}'(m_k) = \mathcal{A}'_{\text{per}}(m_k) = \left( \text{proj } \lim_{n \to \infty} \mathcal{A}_{\text{per}}(m_k, 1/n) \right) = \text{ind lim } \mathcal{A}'_{\text{per}}\left( m_k, \frac{1}{n} \right),
\]

in the sense of strong topologies in these spaces. It means that if \( f \in \mathcal{B}'(m_k) \) then \( f \) can be uniquely extended on some \( \mathcal{A}_{\text{per}}(m_k, 1/n) \) such that \( f \in \mathcal{A}'_{\text{per}}(m_k, 1/n) \). Conversely, if \( f \in \mathcal{A}'_{\text{per}}(m_k, 1/n) \) then \( f \in \mathcal{A}'_{\text{per}}(m_k) \). Thus, it is enough to give a representation theorem for elements from \( \mathcal{A}'_{\text{per}}(m_k, 1/n) \) for a fixed \( n \). This can be done in the same way as in [4, proof of Theorem 6]; see also [3].

The space \( \mathcal{A}_{\text{per}}(m_k, 1/n) \) is isometric to the closed subspace \( \Lambda \) of the space \( (\Gamma, \| \cdot \|_{\Gamma}) \) where \( \Gamma \) is the space of sequences \( \{ \psi_i \} \in \prod_{i=0}^{\infty} L^2(0, 2\pi) \) such that

\[
\| \{ \psi_i \} \|_{\Gamma} := \sum_{i=0}^{\infty} \| \psi_i \|_2 < \infty.
\]

This isometry is given by the mapping

\[
\mathcal{A}_{\text{per}}\left( m_k, \frac{1}{n} \right) \ni \phi \mapsto \left( \frac{(-1)^i n_i \phi^{(i)}}{m_i} \right) \in \Lambda.
\]

Clearly \( f \) defines an element from \( \Lambda' \) which we shall denote by \( f_1 \). By the Hahn Banach theorem we extend \( f_1 \) from \( \Lambda \) to \( \Gamma \) (we denote this extension by \( F \)) such that \( F \in \Gamma' \). Since \( F \in \Gamma' \), there exists a sequence \( f_i, i = 0, 1, \ldots, \) from \( L^2(0, 2\pi) \) such that

\[
\langle F, \{ \psi_i \} \rangle = \sum_{i=0}^{\infty} \langle f_i, \psi_i \rangle = \sum_{i=0}^{\infty} \int_{0}^{2\pi} f_i(t) \psi_i(t) \, dt, \quad \{ \psi_i \} \in \Gamma,
\]

and \( \sup_{i} \| f_i \|_2 < \infty \). If \( \phi \in \mathcal{A}_{\text{per}}(m_k, 1/n) \) we have

\[
\langle f, \phi \rangle = \left( f_1, \left\{ \frac{(-1)^i n_i \phi^{(i)}}{m_i} \right\} \right) = \left( F, \left\{ \frac{(-1)^i n_i \phi^{(i)}}{m_i} \right\} \right) = \sum_{i=0}^{\infty} \left( \frac{n_i}{m_i} f_i \right)^{(i)}, \phi.
\]

This completes the proof of the first part of Theorem 2.

The second part of Theorem 2 trivially holds because any function \( F \) from \( L^2(0, 2\pi) \) defines an element from \( \mathcal{B}'(m_k) \):

\[
\phi \mapsto \int_{0}^{2\pi} F(t) \phi(t) \, dt, \quad \phi \in \mathcal{B}(m_k).
\]
Theorem 3. If \( f \in \mathcal{D}'(m_k) \) then there exists a sequence of functions \( f_i, i = 0, 1, \ldots \), on \((0, 2\pi)\) such that
\[
\begin{align*}
(i) & \quad f_i \in L^2(0, 2\pi); \\
(ii) & \quad \sum_{i=0}^{\infty} n^i m_i ||f_i||_2 < \infty, \text{ for every natural number } n; \\
(iii) & \quad f(t) = \sum_{i=0}^{\infty} f_i^{(i)}(t).
\end{align*}
\]

Conversely, if \( f_i \) is a sequence of functions on \((0, 2\pi)\) such that (i) and (ii) hold, then with the series on the right side of (iii) a unique element from \( \mathcal{D}'(m_k) \) is defined.

The convergence of the series in (iii) is understood in the weak sense.

In the proof which follows, we shall show that \( \mathcal{A}_{\text{per}}(m_k) \) is an inductive limit of an injective compact sequence of \((B)\)-spaces. It implies that \( \mathcal{D}'(m_k) \) and \( \mathcal{D}(m_k) \) are Montel spaces. So the weak and strong sequential convergence in \( \mathcal{D}'(m_k) \) are coincident.

Proof. We denote by \( \mathcal{A}_{\text{per}}(m_k, n) \), where \( n \) is a fixed natural number, a subset of \( \mathcal{A}_{\text{per}} \) such that \( \phi \in \mathcal{A}_{\text{per}}(m_k, n) \) iff \( ||\phi||_{n,2} < \infty \).

Obviously, this space is a \((B)\)-space. As in the proof of Theorem 2, for any \( n_0 \) one can find \( n_3 \) such that the inclusion mapping \( \mathcal{A}_{\text{per}}(m_k, n_0) \rightarrow \mathcal{A}_{\text{per}}(m_k, n_3) \) is compact. Thus we have [3, p. 33]
\[
\mathcal{D}'(m_k) = \mathcal{A}'_{\text{per}}(m_k) = \left( \text{ind lim}_{n \rightarrow \infty} \mathcal{A}_{\text{per}}(m_k, n) \right)'.
\]

We denote by \( Y_n \), where \( n \) is a fixed number, the space of sequences \( \{\psi_i\}, \psi_i \in L^2(0, 2\pi), i = 0, 1, \ldots \), such that
\[
\{\psi_i\} \in Y_n \quad \text{iff} \quad \|\{\psi_i\}\|_{Y_n} := \sum_{i=0}^{\infty} n^i m_i ||\psi_i||_2.
\]
Spaces \( Y_n, n = 1, 2, \ldots \), are \((B)\)-spaces. \( \mathcal{A}_{\text{per}}(m_k, n) \) is isometric with the closed subspace \( \Lambda \) of \( Y_n \). This isometry is given by the mapping \( \mathcal{A}_{\text{per}}(m_k, n) \in \phi \mapsto \{(1)^{\phi(i)}\} \in \Lambda \).

We denote \( \mathcal{A}_{\text{per}}(m_k, n) \) by \( X_n, n = 1, 2, \ldots \). In the same way as in [3, proof of Proposition 8.6] we prove that \( Y_n \) is an injective weakly compact sequence. Since \( X_n \) is an injective compact sequence, \( Z_n = Y_n/X_n \) is also an injective sequence of \((B)\)-spaces. Thus, it follows from the dual Mittag-Leffler lemma [3, p. 37] that if \( f \in (\text{ind lim}_{n \rightarrow \infty} \mathcal{A}_{\text{per}}(m_k, n))' \) then there exists \( F \in (\text{ind lim}_{n \rightarrow \infty} Y_n)' \) such that on \( \Lambda \) \( \langle f, \phi \rangle = \langle F, \{(1)^{\phi(i)}\} \rangle \), and the mapping \( f \mapsto F \) is continuous under the strong topologies in \( \mathcal{D}'(m_k) \) and \( \text{ind lim}_{n \rightarrow \infty} Y_n \).

Since \( \text{ind lim}_{n \rightarrow \infty} Y_n = \text{proj lim}_{n \rightarrow \infty} Y_n \) and this is the space of all sequences \( f_i, f_i \in L^2(0, 2\pi), i = 0, 1, \ldots \), such that for every natural number \( n \sum_{i=0}^{\infty} n^i m_i ||f_i||_2 < \infty \), we obtain that this holds for \( F \).

If \( \phi \in \mathcal{A}_{\text{per}}(m_k) \) we have
\[
\langle f, \phi \rangle = \sum_{i=0}^{\infty} \langle f_i, (1)^{\phi(i)} \rangle = \sum_{i=0}^{\infty} \langle f_i^{(i)}(t), \phi(t) \rangle.
\]
or
\[ f = \sum_{i=0}^{\infty} f^{(i)} \] in \( \mathcal{D}'(m_k) \).

Thus we proved the first part of Theorem 3.

The strong topology in \( \mathcal{D}'(m_k) \) is determined by the family of seminorms
\[ p_n(f) = \sup \{ |\langle f, \phi \rangle|; \|\phi\|_{n,2} < 1 \}, \quad n = 1, 2, \ldots. \]

If \( \phi \) belongs to the unit ball in \( \mathcal{A}_{per}(m_k, n) \) then
\[
|\langle f, \phi \rangle| = \left| \sum_{i=0}^{\infty} \langle f_i, (-1)^i \phi^{(i)} \rangle \right| \leq \sum_{i=0}^{\infty} \int_{0}^{2\pi} |f_i| |\phi^{(i)}| \leq \|n^i m_n f_i\|_2 \|\phi\|_{n,2} \leq C \|\phi\|_{n,2}.
\]

It implies that \( p_m(f) \leq C \) so by the series in (iii) a unique element of \( \mathcal{D}'(m_k) \) is defined.

Let us remark that in the case \( m_k = k! \) the proofs of Theorems 2 and 3 are trivial.


REFERENCES


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