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SOME CLASSES OF ORTHOGONAL POLYNOMIALS ASSOCIATED WITH MARTINGALES

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ABSTRACT. The classes of orthogonal polynomials which arise as iterated stochastic integrals of a process with stationary independent increments are discussed. They are classes of Meixner polynomials.

This study was motivated by the well-known fact [7] that the iterated stochastic integrals of Brownian motion are given by Hermite polynomials $H_n(B(t), t)$. The question arises whether a similar phenomenon holds for other processes with stationary independent increments—do their iterated stochastic integrals yield classes of orthogonal polynomials?

There are two main components involved in the answer. One aspect is the theory of stochastic integrals—work inspired by Meyer [**10**, especially pp. 303–307, 318–320]. Two theorems are critical:

(1) LIN'S THEOREM [**8**]. *The generating function for the iterated stochastic integrals is the exponential martingale of Doléans-Dade.*

(2) EMERY'S THEOREM (e.g. [**4**]). *In the continuous case, the exponential martingale can be computed via a suitable discretization procedure.* (See his Theorem 2, p. 261, with $e(x) = 1 + vx$, cf. below.)

The second aspect is the intimate relationship of the Meixner polynomials with the representation theory of the Heisenberg group (see [**5, 6**]). The main feature here is that the lowering (pseudo-differential) operator $V(d/dx)$ satisfies a Riccati equation (cf. [**5, 9**]). The lowering operator V acts on the polynomial sequence $J_n(x)$ by $VJ_n = nJ_{n-1}$ and the polynomials J_n are given as $\xi^n 1$, where V, ξ form a canonical pair of boson operators, i.e., their commutator $[V, \xi] = 1$. In other words, the class of orthogonal polynomials arising in this study are identical to those that arise as Fock spaces in one variable—the function 1 plays the role of the vacuum Ω, V is the annihilation operator, ξ the creation operator (cf. [**6**]).

Thanks to the theorems mentioned above, we consider the discrete case first, then take limits to deal with the continuous case. We begin with a probability measure

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$p(dx)$ on $\mathbf{R}$ satisfying

$$\int_{\mathbf{R}} e^{i\xi x} p\,(dx) = e^{L(i\xi)}, \qquad \xi \in \mathbf{R},$$

where we assume that the function L has an analytic extension to a neighborhood of 0 in $\mathbf{C}$. We denote this extension by $L(z)$, $z \in \mathbf{C}$. E.g., if $p(dx) = (\exp(-x^2/2)/\sqrt{2\pi})\,dx$, then $L(z) = z^2/2$. The convolution powers of p are denoted by p_t, where in the discrete case $t \in \mathbf{N}$. In the continuous case L must be of Lévy-Khinchine type; no such restriction applies in the discrete case. (Note. In the sequel we use the notation $p(dx) = p(x)\,dx$, in the sense of distributions.)

I. Rodrigues formulae. In this case we consider the process $S_n = X_1 + X_2 + \cdots + X_n$, where the X_j are independent, mean zero, with distribution p. The exponential martingale is

$$E_n(v) = \prod_{j=1}^{n} (1 + vX_j) = \sum_{k=0}^{n} v^k I_k.$$

The I_k are discrete iterated sums, readily recognized as the elementary symmetric functions in the variables $X_1, X_2, \ldots, X_n$. The calculation, $\langle\ \ \rangle$ denoting expected value, $\sigma^2 = \langle X_j^2 \rangle$,

$$\langle E(v)E(w)\rangle = \prod_{1}^{n} \langle 1 + (v + w)X_j + vwX_j^2 \rangle = (1 + vw\sigma^2)^n$$

shows that the I_k are orthogonal, the cross terms vanishing.

In order to have orthogonal polynomials, these I_k must be functions of the variable x, that is, of S_n alone (besides n, of course). If this is the case we say that the process S_n is *observable*—since the I_k are functions only of the "physical parameters" $x = S_n$ and $t = n$, space and time.

THEOREM 1. *The exponential martingale for an observable discrete process is of the form*

$$\mu(x,t) = p_t(x)^{-1}(1 + vV^*)^t p_t(x)$$

where the pseudo-differential operator $V^*(D) = V(-D)$ where V has symbol $V(z) = L'(z)$.

REMARK. The equality stated means that $\mu(x,t)p(x) = (1 + vV^*)^t p(x)$ in the sense of distributions. (Note. $D = d/dx$.)

PROOF. The key notion is that for the observable case since $E_n(v)$ is only a function of $x = S_n$ and $t = n$ it must equal the conditional expectation

$$\mu(x,t) = E(E_n(v)|S_n = x) = \langle E_n\delta(S_n - x)\rangle / p_n(x).$$

Using the Fourier representation $\delta(y) = \int_{\mathbf{R}} e^{i\xi y}\,d\xi/2\pi$,

$$\mu(x,t)p_t(x) = \int \left\langle \prod_j (1 + vX_j)e^{i\xi X_j} \right\rangle e^{-i\xi x}\,d\xi/2\pi.$$

Consider a single factor with X replacing an X_j,

$$\begin{aligned}\langle (1+vX)e^{i\xi X}\rangle &= \left(1+\frac{v}{i}\frac{\partial}{\partial\xi}\right)\langle e^{i\xi X}\rangle \\ &= \left(1+\frac{v}{i}\frac{\partial}{\partial\xi}\right)e^{L(i\xi)} \\ &= (1+vL'(i\xi))e^{L(i\xi)}.\end{aligned}$$

Thus, with D denoting d/dx,

$$\begin{aligned}\mu(x,t)p_t(x) &= \int_{\mathbf{R}} (1+vL'(i\xi))^n e^{nL(i\xi)}e^{-i\xi x}\,dx/2\pi \\ &= (1+vL'(-D))^n \int_{\mathbf{R}} e^{-i\xi x}e^{nL(i\xi)}\,d\xi/2\pi \\ &= (1+vV^*)^t p_t(x)\end{aligned}$$

as required.

THEOREM 2. *The exponential martingale for an observable continuous process is of the form*

$$\mu(x,t) = p_t(x)^{-1}e^{vtV^*}p_t(x)$$

with $V(z) = L'(z)$.

PROOF. By the theorems mentioned above, we consider the process $S(t) = X_1 + X_2 + \cdots + X_n$ where the X_j are independent, mean zero, with distribution $p_{t/n}$, i.e.

$$X_j = S(jt/n) - S((j-1)t/n), \qquad S(0) = 0.$$

The discrete approximant satisfies

$$\mu_n(x,t) = \langle \mu_n\delta(S(t)-x)\rangle/p_t(x).$$

As above, we find

$$\mu_n(x,t)p_t(x) = (1+vtV^*/n)^n p_t(x).$$

In the limit as $n \to \infty$ we arrive at the stated form.

Now, expanding the generating functions we have the iterated stochastic integrals expressed via the Rodrigues-type formulae

$$I_k(x,t) = c_k p_t(x)^{-1}V^{*k}p_t(x)$$

with suitable constants c_k.

II. Riccati equation. The next step is to require that the I_k be in fact polynomials in x, t.

LEMMA 1. *Let* $\eta_k(z) = e^{-tL(z)}(\partial/\partial z)^k e^{tL(z)}$. *Then* $\eta_k(i\xi)e^{tL(i\xi)}$ *is the Fourier transform of* $x^k p_t(x)$. *That is, as operators, the* η_k *act according to* $\eta_k^* p_t(x) = x^k p_t(x)$.

PROOF. Taking Fourier transforms, using the standard relations

$$(x^k f(x))\check{}(\xi) = \left(\frac{1}{i}\frac{\partial}{\partial\xi}\right)^k f\check{}(\xi)$$

yields the result, with $f\check{}(\xi) = \int_{\mathbf{R}} e^{i\xi x}f(x)\,dx$.

THEOREM 3. *For the $I_k(x,t)$ to be polynomials, the function $V(z)$ must satisfy a differential equation of the form $V' = a + bV + cV^2$ for suitable constants a, b, c.*

PROOF. For $k = 1$, $\eta_1(z) = tL'(z) = tV(z)$. For $k = 2$, $\eta_2(z) = tL''(z) + (tL'(z))^2 = \alpha V' + \beta V^2$, say. The polynomial $I_2(x,t)$ satisfies

$$cI_2(x,t)p_t(x) = V^{*2}p_t(x)$$

or, by the lemma, there exist constants λ, μ, ν, and λ_j such that

$$V^2 = \lambda + \mu\eta_1 + \nu\eta_2 = \lambda_1 + \lambda_2 V + \lambda_3 V' + \lambda_4 V^2.$$

If $\lambda_3 = 0$, then the function $V(z)$ would satisfy a quadratic equation. The centering of the process gives $V(0) = L'(0) = 0$ and thus V would be constant. Hence $\lambda_3 \neq 0$ and the result follows.

REMARK. Notice that only I_1 and I_2 play an explicit role, cf. [**1**].

III. Conclusion. It now follows that the polynomials are in fact Meixner polynomials (cf. [**2, 6, 9**]). That is, the processes are of the types: binomial, negative binomial, gamma, Poisson, or Gaussian. Briefly, we may say these are of "Bernoulli type."

It is interesting to see how in the Bernoulli case the two-valuedness leads to observability. Consider X_j taking values 1 and 0, $P(X_j = 1) = p$. Let ν = number of X_j equalling 0, π = number of the X_j equalling $+1$. For the process $\sum_1^n X_j$ at position x at time t, we have $x = \pi$, $t = \pi + \nu$. Form the centered increments $Y_j = X_j - p$. Thus,

$$E_n(v) = \prod_1^n (1 + vY_j) = (1 + v(1-p))^{\pi}(1 - pv)^{\nu}$$

$$= (1 + v(1-p))^{x}(1 - pv)^{t-x}.$$

REMARK. A recent paper by Dunkl [**3**] discusses some Dirichlet problems on the disk. The Meixner-Pollaczek polynomials play a major role. They are of Bernoulli type with $L(z) = \log\cos\beta - \log\cos(z + \beta)$ (see [**3**, p. 170]). They correspond to infinitely divisible laws.

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