BETTER BOUNDS FOR PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. Let \( f \) be Lipschitz with constant \( L \) in a Banach space and let \( x(t) \) be a \( P \)-periodic solution of \( x'(t) = f(x(t)) \). We show that \( P \geq 6/L \). An example is given with \( P = 2\pi/L \), so the bound is nearly strict. We also give a short proof that \( P \geq 2\pi/L \) in a Hilbert space.

I. Introduction. Suppose \( f \) is Lipschitz with constant \( L \) in a Banach space and \( x(t) \) is a \( P \)-periodic solution of \( x'(t) = f(x(t)) \). How small can \( LP \) be?

This question was first addressed when Lasota and Yorke\cite{3} proved \( LP \geq 4 \). Busenberg and Martelli\cite{2} showed \( LP \geq 4\frac{1}{2} \). We show that \( LP \geq 6 \).

The lowest known \( LP \) is \( 2\pi \), e.g. for nonzero solutions of \( u'(t) = v(t) \) and \( v'(t) = -u(t) \). Also, in a Hilbert space, \( LP \geq 2\pi \) (Lasota and Yorke\cite{3}, Yorke\cite{4}—A short proof is given in \S III). So the Hilbert space bound is strict, but the Banach space bound may not be strict.

II. In Banach spaces, \( LP \geq 6 \).

LEMMA 1. Let \( B \) be a Banach space and let \( y: \mathbb{R} \rightarrow B \) be continuous and \( P \)-periodic with \( ||y'(t)|| \) integrable. Then

\[ \int_0^P \int_0^P ||y(t) - y(s)|| \, ds \, dt \leq \frac{P}{6} \int_0^P \int_0^P ||y'(t) - y'(s)|| \, ds \, dt. \]

PROOF.

\[ A = \int_0^P \int_0^P ||y(t) - y(s)|| \, ds \, dt = \int_0^P \int_0^P ||y(s + t) - y(s)|| \, ds \, dt \]

\[ = \int_0^P \int_0^P \frac{(P - t)t}{P} \left( \frac{y(s + t) - y(s)}{t} - \frac{y(s) - y(s + t - P)}{P - t} \right) \, ds \, dt \]

\[ = \int_0^P \int_0^P \frac{(P - t)t}{P^2} \left( \int_0^P \left( y' \left( s + \frac{tr}{P} \right) - y' \left( s + \frac{tr - r}{P} \right) \right) \, dr \right) \, ds \, dt \]

\[ \leq \int_0^P \int_0^P \frac{(P - t)t}{P^2} \int_0^P \left( y' \left( s + \frac{tr}{P} \right) - y' \left( s + \frac{tr - r}{P} \right) \right) \, dr \, ds \, dt \]

\[ = \int_0^P \int_0^P \frac{(P - t)t}{P^2} \int_0^P \left( y' \left( s + \frac{tr}{P} \right) - y' \left( s + \frac{tr - r}{P} \right) \right) \, dr \, ds \, dt. \]
Since the inner integral is over one period, it can be shifted by \( tr/P - r \) to yield
\[
A < \int_0^P \frac{(P - t)^2}{P^2} dt \int_0^P \left\| y'(s + r) - y'(s) \right\| ds \, dr
\]
\[
= \frac{P}{6} \int_0^P \int_0^P \left\| y'(r) - y'(s) \right\| ds \, dr.
\]

**Theorem 1.** If \( f \) is Lipschitz with constant \( L \) in a Banach space and \( x(t) \) is a nonconstant \( P \)-periodic solution of \( x'(t) = f(x(t)) \), then \( LP \geq 6 \).

**Proof.**
\[
\int_0^P \int_0^P \left\| x(t) - x(s) \right\| ds \, dt \leq \frac{P}{6} \int_0^P \int_0^P \left\| x'(t) - x'(s) \right\| ds \, dt
\]
\[
= \frac{P}{6} \int_0^P \int_0^P \left\| f(x(t)) - f(x(s)) \right\| ds \, dt \leq \frac{LP}{6} \int_0^P \int_0^P \left\| x(t) - x(s) \right\| ds \, dt.
\]
Solving for \( LP \) gives the result.

**III. A short proof that **\( LP \geq 2\pi \) **in Hilbert spaces.**

**Lemma 2 (a Hilbert space analog of Wirtinger’s inequality).** Let \( H \) be a Hilbert space and let \( y: \mathbb{R} \to H \) be continuous and \( P \)-periodic with \( \int_0^P y(t) \, dt = 0 \) and \( \| y'(t) \|^2 \) integrable. Then
\[
\int_0^P \| y(t) \|^2 \, dt \leq \frac{P^2}{4\pi^2} \int_0^P \| y'(t) \|^2 \, dt.
\]

**Proof.** Since the path of \( y(t) \) is compact, there is a countable orthonormal set \( e_1, e_2, \ldots \) with \( y(t) = \sum_{i=1}^{\infty} a_i(t) e_i \). Each \( a_i(t) \) is \( P \)-periodic and \( \int_0^P a_i(t) \, dt = 0 \), so Wirtinger’s inequality [1] gives \( \int_0^P a_i(t)^2 \, dt \leq (P^2/4\pi^2) \int_0^P a_i'(t)^2 \, dt \). Then
\[
\int_0^P \| y(t) \|^2 \, dt = \int_0^P \left\| \sum_{i=1}^{\infty} a_i(t) e_i \right\|^2 \, dt = \int_0^P \sum_{i=1}^{\infty} a_i(t)^2 \, dt
\]
\[
= \sum_{i=1}^{\infty} \int_0^P a_i(t)^2 \, dt \leq \frac{P^2}{4\pi^2} \sum_{i=1}^{\infty} \int_0^P a_i'(t)^2 \, dt = \frac{P^2}{4\pi^2} \int_0^P \sum_{i=1}^{\infty} a_i'(t)^2 \, dt
\]
\[
= \frac{P^2}{4\pi^2} \int_0^P \left\| \sum_{i=1}^{\infty} a_i'(t) e_i \right\|^2 \, dt = \frac{P^2}{4\pi^2} \int_0^P \| y'(t) \|^2 \, dt.
\]

**Theorem 2 [3, 4].** If \( f \) is Lipschitz with constant \( L \) in a Hilbert space and \( x(t) \) is a nonconstant \( P \)-periodic solution of \( x'(t) = f(x(t)) \), then \( LP \geq 2\pi \).

**Proof.** Pick \( h \) with \( x(h) \neq x(0) \). Since \( x(t) \) is \( P \)-periodic, \( x(t + h) - x(t) \) is \( P \)-periodic and \( \int_0^P (x(t + h) - x(t)) \, dt = 0 \). Then from Lemma 2
\[
\int_0^P \| x(t + h) - x(t) \|^2 \, dt \leq \frac{P^2}{4\pi^2} \int_0^P \| x'(t + h) - x'(t) \|^2 \, dt
\]
\[
= \frac{P^2}{4\pi^2} \int_0^P \| f(x(t + h)) - f(x(t)) \|^2 \, dt \leq \frac{L^2 P^2}{4\pi^2} \int_0^P \| x(t + h) - x(t) \|^2 \, dt.
\]
Solving for \( LP \) gives the result.

**Note added in proof.** The bound in Theorem 1 is now known to be sharp.
We have constructed an example in a Banach space with \( LP = 6 \).
REFERENCES


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