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ON COMPARABILITY IN A TOPOS

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Dedicated to EJA

ABSTRACT. While studying the category of "finite" sets associated to a nonstandard model of arithmetic [4], it became apparent that the law of trichotomy plays an important role in that context. The object of this note is to point out the strength of the obvious variants of trichotomy in a general topos. At it turns out even its mildest form (COMP) is quite restrictive, the usual variants (M-COMP, E-COMP) force the topos to be equivalent to a category of sets with AC.

DEFINITION. A topos $\mathbf{E}$ satisfies **comparability** (COMP) if for any two objects A, B there is a morphism $A \to B$ or $B \to A$.

Observations. a. If $\mathbf{E}$ satisfies COMP, so does any full subcategory of $\mathbf{E}$, in particular $SH_j(\mathbf{E})$ for some topology j, and $\mathbf{E}_G$, the topos of coalgebras for an idempotent left exact cotriple G.

b. Any filtered colimit of toposes with COMP has COMP.

c. In a topos with COMP the lattice of subobjects of 1 is totally ordered.

d. COMP is not stable ($\mathbf{S}$, the category of sets with AC, has COMP, but $\mathbf{S}/A$ doesn't).

PROPOSITION 1. *If supports split in* $\mathbf{E}$ *and the subobjects of* 1 *are totally ordered, then* $\mathbf{E}$ *has* COMP.

PROOF. Given A, B we have epimorphisms to their supports $A \twoheadrightarrow \sigma(A)$, $B \twoheadrightarrow \sigma(B)$, also their splittings going in the opposite direction, as well as, say $\sigma(A) \rightarrowtail \sigma(B)$, thus by composition, a morphism from A to B.

EXAMPLES. It is well known that if $\mathbf{S}$ satisfies AC then COMP is true in $\mathbf{S}$. For any well order T, $\mathbf{S}^T$ has COMP. Let M be the monoid consisting of e and a single idempotent element a, then $\mathbf{S}^{M^{\mathrm{op}}}$ has COMP. In both cases the conditions of Proposition 1 are fulfilled. Let G be a group whose lattice of subgroups divided by the conjugation relation is a total order (e.g. $Z/(p^n)$); then $\mathbf{S}^G$ satisfies COMP.

PROPOSITION 2. *If* $\mathbf{E}$ *is a Grothendieck topos and has* COMP, *then* $\mathbf{E}$ *is connected and locally connected* (*molecular*).

PROOF. Indeed by c, there are no nontrivial complemented subobjects of $\mathbf{1}$, i.e., $\mathbf{E}$ is connected. Let $\mathbf{C}$ be a generating full subcategory of $\mathbf{E}$; then by Giraud's theorem

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we have $\mathbf{E} \approx Sh_{\text{can}}(\mathbf{C})$. The canonical sieves are nonempty and COMP insures that they are connected. Hence $(\mathbf{C}, \text{can})$ is a molecular site (see [**1**]). ■

DEFINITION. A topos $\mathbf{E}$ satisfies **mono-comparability** (M-COMP) if for any two objects A, B there is a monomorphism $A \rightarrowtail B$ or $B \rightarrowtail A$.

PROPOSITION 3. *If a topos satisfies* M-COMP, *then supports split.*

PROOF. Let $p\colon A \twoheadrightarrow \sigma(A)$ be the support morphism. If there is a monomorphism $m\colon A \rightarrowtail \sigma(A)$, then the unicity of morphisms to $\sigma(A)$ implies that $p = m$. Hence it has an inverse. On the other hand, if there is a monomorphism $m\colon \sigma(A) \rightarrowtail A$, then, again by the same unicity, we must have $pm = \text{id}_{\sigma(A)}$. ■

PROPOSITION 4. *In an* M-COMP *topos the terminal object* $\mathbf{1}$ *has no proper subobjects.*

PROOF. Let $U \rightarrowtail \mathbf{1}$ be a subobject; then if the unique morphism $!\colon U + U \to \mathbf{1}$ is a monomorphism, then from $!i_1 = !i_2$ it follows that $i_1 = i_2$ hence $U = \text{eq}(i_1, i_2) \approx \mathbf{0}$. Otherwise a monomorphism $\mathbf{1} \rightarrowtail U + U$ implies $\mathbf{1} \approx \sigma(U + U) \approx \sigma(U) \approx U$. ■

PROPOSITION 5. *An* M-COMP *topos is Boolean.*

PROOF. By Proposition 4 the only morphisms $\mathbf{1} \rightarrowtail \Omega$ are **true** and **false**. This implies that there is no monomorphism from $\mathbf{1} + \mathbf{1} + \mathbf{1}$ to Ω. So there must be a monomorphism $\Omega \rightarrowtail \mathbf{1} + \mathbf{1} + \mathbf{1}$ which splits Ω into three summands $\Omega \approx T + F + X$. T and F are respectively **true** and **false** because we have in any topos the canonical monomorphism $\mathbf{1} + \mathbf{1} \rightarrowtail \Omega$. Now, if X is not isomorphic to $\mathbf{0}$, then the morphism $X \to \mathbf{1}$ must be an epimorphism, and by Proposition 3 it has a section $\mathbf{1} \rightarrowtail X$ which provides a third morphism from $\mathbf{1}$ to Ω, contradicting the first statement of this proof. ■

PROPOSITION 6. *In a topos with* M-COMP $\mathbf{1}$ *is a generator.*

PROOF. Let $u, v\colon x \rightrightarrows Y$ be any two morphisms and let E be their equalizer, which by Proposition 5 admits a complement E'. If E' is not $\mathbf{0}$, then, as above, there is a morphism $x\colon \mathbf{1} \rightarrowtail E'$, whence $ux \neq vx$. ■

DEFINITION. A topos $\mathbf{E}$ satisfies **epi-comparability** (E-COMP) if for any objects $A, B \neq 0$ there exists an epimorphism $A \twoheadrightarrow B$ or $B \twoheadrightarrow A$.

PROPOSITION 7. *In an* E-COMP *topos the terminal object* $\mathbf{1}$ *has no proper subobjects.*

PROOF. Let $U \rightarrowtail \mathbf{1}$ be a subobject and, again, one looks at $U + U$. If the unique morphism $U + U \to \mathbf{1}$ is an epimorphism, then so must be $U \to \mathbf{1}$ (since $\sigma(U + U) = \sigma(U)$), hence it is an isomorphism. If there is an epimorphism $\mathbf{1} \twoheadrightarrow U + U$, then passing to supports yields $\mathbf{1} = \sigma(\mathbf{1}) \rightarrowtail \sigma(U + U) \approx \sigma(U) \approx (U)$. ■

The following is a topos version of a celebrated result due to G. Cantor:

LEMMA 8. *Let* $(\Delta, \Gamma)\colon E \to S$ *be any topos defined over sets and* $E \in \mathbf{E}$. *Then there are no epimorphisms* $E \twoheadrightarrow \Delta\Gamma(\mathbf{2}^E)$.

PROOF. Assume that there exists an epimorphism $p\colon E \twoheadrightarrow \Delta\Gamma(\mathbf{2}^E)$ and use it to form the morphism:

$$E \underset{(p,E)}{\rightarrowtail} \Delta\Gamma(\mathbf{2}^E) \times E \underset{(\varepsilon \times E)}{\to} \mathbf{2}^E \times E \underset{ev}{\to} \mathbf{2}$$

To it corresponds a complemented subobject $m\colon E' \rightarrowtail E$ with characteristic morphism χ; let $\varphi = \neg\, \chi$, with associated subobject $\mathbf{C}_E E'$, and $\ulcorner\varphi\urcorner\colon \mathbf{1} \rightarrowtail \mathbf{2}^E$ its exponential transpose. Applying $\Delta\Gamma$, one obtains the morphism $\Delta\Gamma(\ulcorner\varphi\urcorner)\colon \mathbf{1} \rightarrowtail \Delta\Gamma(\mathbf{2}^E)$ satisfying the usual $\varepsilon\Delta\Gamma(\ulcorner\varphi\urcorner) = \ulcorner\varphi\urcorner$. Pulling back along pm one obtains:

$$\begin{array}{ccccc} F' & \rightarrowtail & F & \twoheadrightarrow & \mathbf{1} \\ \downarrow n' & & \downarrow n & & \downarrow \Delta\Gamma(\ulcorner\varphi\urcorner) \\ E' & \overset{m}{\rightarrowtail} & E & \overset{p}{\twoheadrightarrow} & \Delta\Gamma(\mathbf{2}^E) \end{array}$$

Note that $F \neq \mathbf{0}$. From the commutativity of the rightmost square one obtains, after composing with $ev(\varepsilon \times E)$, two equal morphisms. On one hand, by unraveling the definition of $\Delta\Gamma(\ulcorner\varphi\urcorner)$, one has a morphism $F \times E \to E \to \mathbf{2}$, whose associated subobject is: $F \times \mathbf{C}_E E' \rightarrowtail F \times E$. The pull-back of it along (F, n) is $\mathbf{C}_F F'$. On the other hand, one has a morphism which composed with (F, n) yields the commutative diagram:

$$\begin{array}{ccccccccc} F & \overset{n}{\rightarrowtail} & E & & & & & & \\ \downarrow (F,n) & & \downarrow (E,E) & \searrow (p,E) & & & & & \\ F \times E & \overset{n\times E}{\rightarrowtail} & E \times E & \overset{p\times E}{\twoheadrightarrow} & \Delta\Gamma(\mathbf{2}^E) \times E & \overset{\varepsilon\times E}{\to} & \mathbf{2}^E \times E & \overset{ev}{\to} & \mathbf{2} \end{array}$$

The associated subobject of F is none other than F'. Therefore $\mathbf{C}_F F' \approx F'$, a contradiction since $F \neq \mathbf{0}$. ■

COROLLARY 9. *For any topos* $\mathbf{E}$ *and for any* $E \in \mathbf{E}$ *there exists* $S \in \mathbf{S}$ *such that there are no epimorphisms from* E *to* $\Delta(S)$. ■

PROPOSITION 10. *An* E-COMP *topos is Boolean.*

PROOF. Consider Ω and ΔS, for arbitrarily large S. The preceding corollary implies the existence of an epimorphism $p\colon \Delta S \twoheadrightarrow \Omega$. As there are only two morphisms from $\mathbf{1} = \Delta\mathbf{1}$ to Ω, p must factor through the canonical monomorphism $\Delta\mathbf{2} = \mathbf{1} + \mathbf{1} \rightarrowtail \Omega$, which forces it to be an epimorphism as well, hence an isomorphism. ■

PROPOSITION 11. *In a topos with* E-COMP $\mathbf{1}$ *is a generator.*

PROOF. First, $\mathbf{E}$ is molecular, so let G be a generator which is a molecule as well. As before, there is $S \in \mathbf{S}$ and an epimorphism $p\colon \Delta S \twoheadrightarrow G$. Any element $s\colon \mathbf{1} \rightarrowtail S$ induces a monomorphism $p\Delta(s)\colon \mathbf{1} \rightarrowtail G$ which has a complement G'. G being a molecule forces $G' \approx \mathbf{0}$. Thus $p\Delta(s)$ is an isomorphism. ■

In view of their similar consequences, one suspects that M-COMP and E-COMP are equivalent for any topos (they are in set theory [6]). Indeed, this is the case, but these consequences are used in the present proof. First, let us take a diversion through well-ordered objects in a Boolean topos in which $\mathbf{1}$ is a generator. In such a topos $\Omega = \mathbf{2}$ and let $\mathbf{P}(A) = \mathbf{2}^A$ and $\mathbf{P}_*(A) = \mathbf{P}(A) - \{\mathbf{false}\}$, i.e., the object of nonempty subobjects of A.

Given a partial order $\leqslant \rightarrowtail A \times A$ one derives a morphism $[\ \uparrow)$: $A \to \mathbf{P}(A)$, $[a\uparrow) = \{x\colon a \leqslant x\}$, which factors through $\mathbf{P}_*(A)$.

Definition. $(A, \leqslant)$ is well ordered if there exists a morphism min: $\mathbf{P}_*(A) \to A$ such that (i) $\min[a\uparrow) = a$, (ii) $X \subset [\min(X)\uparrow)$, and (iii) $\min(X) \in X$.

Lemma 12. *Let A be a well-ordered object and $B \not\subset A$ an initial segment. Then there is $b \in A$ such that $B = (b\downarrow) = \{x\colon x < b\}$.*

Proof. Set $b = \min(\mathbf{C}_A B)$. ■

Lemma 13. *Given two well-ordered objects $(A, \leqslant), (B, \leqslant)$, there is an order-preserving monomorphism $m\colon A \rightarrowtail B$, or $n\colon B \rightarrowtail A$, which is initial.*

Proof (adapted from Zermelo via [2]). Let Φ be the object of elements $\varphi \subset \mathbf{P}(A \times B)$ that satisfy the conditions: (i) $\mathbf{0} \in \varphi$, (ii) $\varphi' \subset \varphi$ implies $\bigcup\varphi' \in \varphi$, (iii) $X \in \varphi$ and $p_1 X \neq A$ and $p_2 X \neq B$ implies $X^* \in \varphi$, where $X^* = X \cup \{\min(A - p_1 X), \min(B - p_2 x)\}$. Note that $X \not\subset X^*$. Let $\varphi_0 = \bigcap\{\varphi\colon \varphi \in \Phi\}$ and $C = \bigcup\{X\colon X \in \varphi_0\}$. Thus $C \subset A \times B$ and $(a', b), (a'', b) \in C$ implies $a' = a''$, as well as $(a, b'), (a, b'') \in C$ implies $b' = b''$. Indeed, say $a' < a''$ and let $\varphi' = \{X \in \varphi_0\colon (a'', b) \notin X\}$, φ' satisfies (i) and (ii) trivially and $\varphi' \subset \varphi_0$. If $(a', b) \in X$, then $\min(B - p_2 X) > b$, thus $(a'', b) \notin X^*$; if $(a', b) \notin X$ then $\min(A - p_1 X) \leqslant a'$ so that $(a'', b) \notin X^*$, i.e., in both cases $X^* \in \varphi'$. So φ' satisfies (iii) as well, whence $\varphi_0 = \varphi'$ and $(a'', b) \notin C$. Similarly for the other assertion. Now let $\psi = \{X \in \varphi_0\colon p_1 X$ is initial in $A \wedge p_2 X$ is initial in $B\}$ and let $D = \bigcup\{X\colon X \in \psi\}$. Obviously $D \neq \varnothing$, $D \subset C$ and $p_1 D$, $p_2 D$ are respectively initial in A, B. By (ii) one has $D \in \varphi_0$, so, if $p_1 D \neq A$ and $p_2 D \neq B$ one can construct D^*. Again, $p_1 D^*$, $p_2 D^*$ are initial, hence $D^* \in \psi$, which implies that $D^* \subset D$, a contradiction. It follows that $p_1 D = A$ or $p_2 D = B$, which means that D provides an initial embedding of A into B or vice versa. ■

Hartogs' operator $(\ )^+$ can be defined as well. Given an object X let $X^\sim = \{(A, \leqslant)\colon A \subset X, \leqslant$ is a w.o. on $A\}$ and define $(A', \leqslant') \leqslant_1 (A'', \leqslant'')$ to mean the existence of an order-preserving monomorphism $m\colon A' \rightarrowtail A''$ such that the $\mathrm{Im}(m)$ is an initial segment of A''. $(A', \leqslant') \equiv (A'', \leqslant'')$ means that m is an isomorphism. Let $X^+ = X^\sim/\equiv$, then $\leqslant_1$ induces a well order on X^+. For any nonempty family $\{B_j\}_{j \in J}$ of elements of X^+, choose any $k \in J$ and a representative A_k for B_k. Lemma 13 says that the family $\{B_j\}_{j \in J}$ is totally ordered so one can ignore those B_j with $B_j > B_k$. For each j with $B_j \leqslant B_k$ let A_j be the initial segment of A_k provided by Lemma 13. In turn, for each A_j one has elements a_j given by Lemma 12, forming a subobject of A_k whose min is a_0. Then the corresponding A_0 and B_0 are minimal as well; therefore X^+ is well ordered.

THEOREM 14. *The following are equivalent*:
(a) **E** *satisfies* M-COMP.
(b) **E** *satisfies* E-COMP.
(c) **E** *satisfies* AC *and* **1** *is a generator*.

PROOF. (a) $\Rightarrow$ (b) Let $A, B \neq 0$; by hypothesis there is a monomorphism, say m: $A \rightarrowtail B$; $A \neq 0$ implies that there is a morphism a: $\mathbf{1} \to A$, so one can define an epimorphism p: $B \twoheadrightarrow A$ by cases $p(b) = m^{-1}(b)$ for $b \in \mathrm{Im}(m)$ and $p(b) = a$ for $b \notin \mathrm{Im}(m)$.

(b) $\Rightarrow$ (c) Once the Hartogs' operator is definable, the proof in [**6**, p. 10] is valid in a Boolean topos in which **1** generates.

(c) $\Rightarrow$ (a) It is known that AC forces the topos **E** to be Boolean [**3**], so that the proof outlined in [**2**, ex. 4, p. 201] is completely in **E**. ■

REFERENCES

1. M. Barr and R. Paré, *Molecular toposes*, J. Pure Appl. Algebra **17** (1980), 127–152.
2. D. van Dalen, H. C. Doets and H. de Swart, *Sets*: *naive*, *axiomatic and applied*, Pergamon Press, 1978.
3. R. Diaconescu, *Axiom of choice and complementation*, Proc. Amer. Math. Soc. **51** (1975), 176–178.
4. R. Diaconescu and L. A. S. Kirby, *Models of arithmetic and categories with finiteness conditions*, 1985 (preprint).
5. P. T. Johnstone, *Topos theory*, Academic Press, 1977.
6. H. Rubin and J. E. Rubin, *Equivalents of the axiom of choice*, North-Holland, 1963.

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