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PRIME IDEALS IN ALGEBRAS OF CONTINUOUS FUNCTIONS

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ABSTRACT. Let X_0 be a compact Hausdorff space, and let $\mathbf{C}(X_0)$ be the Banach algebra of all continuous complex-valued functions on X_0. It is known that, assuming the continuum hypothesis, any nonmaximal, prime ideal $\mathbf{P}$ such that $|\mathbf{C}(X_0)/\mathbf{P}| = 2^{\aleph_0}$ is the kernel of a discontinuous homomorphism from $\mathbf{C}(X_0)$ into some Banach algebra. Here we consider the converse question of which ideals can be the kernels of such a homomorphism. Partial results are obtained in the case where X_0 is metrizable.

Introduction. Let X_0 be a compact Hausdorff space, let $\mathbf{C}(X_0)$ be the Banach algebra of all continuous complex-valued functions on X_0, and let Θ be a homomorphism from $\mathbf{C}(X_0)$ into a Banach algebra $\mathbf{B}$. It is an old question whether or not each such homomorphism is automatically continuous: we know that, if the continuum hypothesis (CH) be assumed, there are discontinuous homomorphisms from $\mathbf{C}(X_0)$ for each infinite space X_0 (see [**2**, **8**]), and also that there are models of set theory in which each such homomorphism is continuous [**15**, **4**]. Nevertheless, there remain a number of open questions about these homomorphisms. Some, involving the case where CH does not hold, are discussed in [**4**, Chapter 3]. Here, we consider a question about the kernels of discontinuous homomorphisms when CH does hold.

The seminal study of homomorphisms $\Theta\colon \mathbf{C}(X_0) \to \mathbf{B}$ is that of Bade and Curtis [**1**]. They showed, in particular, that, if Θ is discontinuous, there is a finite, nonempty set $F = \{x_1, \ldots, x_n\}$ of nonisolated points of X_0, a continuous homomorphism $\mu\colon \mathbf{C}(X_0) \to \mathbf{B}$, and linear maps $\lambda_1, \ldots, \lambda_k\colon \mathbf{C}(X_0) \to \mathbf{B}$, such that $B = \mu + \lambda_1 + \cdots + \lambda_k$, such that $\lambda_i | \mathbf{M}_{x_i}$ is a nonzero homomorphism into the radical of $\mathbf{B}$, and such that $\lambda_i | \mathbf{J}_{x_i} = 0$ $(i = 1, \ldots, k)$. Here, $\mathbf{M}_x = \{f\colon f(x) = 0\}$ and $\mathbf{J}_x = \{f\colon f^{-1}(\{0\})$ is a neighborhood of $x\}$ for $x \in X_0$. Since $\mathbf{J}_{x_i}$ is dense in $\mathbf{M}_{x_i}$, λ_i is discontinuous $(i = 1, \ldots, k)$. The set F is the *singularity set* of Θ.

Let Θ be a homomorphism from $\mathbf{C}(X_0)$. Then $q\colon f \mapsto \|\Theta(f)\|$ is a seminorm on $\mathbf{C}(X_0)$, and q is continuous if and only if Θ is continuous. Let $\mathbf{F}_q$ be the set of nonmaximal, prime, q-closed ideals in $\mathbf{C}(X_0)$. It is shown by Esterle in [**5**] that $\mathbf{F}_q = \varnothing$ if and only if q is discontinuous (see also [**13**]). In this case, set $\mathbf{I}_q = \bigcap\{\mathbf{P}\colon \mathbf{P} \in \mathbf{F}_q\}$. Then: (i) the set of maximal ideals of $\mathbf{C}(X_0)$ containing an element of $\mathbf{F}_q$ is finite; (ii) $\Theta | \mathbf{I}_q$ is continuous, and, if $\mathbf{I}$ is an ideal of $\mathbf{C}(X_0)$ such that $\Theta | \mathbf{I}$ is continuous, then $\mathbf{I} \subset \mathbf{I}_q$; (iii) each chain in $\mathbf{F}_q$ is well-ordered with respect to inclusion. This theorem of Esterle extends the earlier result of Bade and Curtis.

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It is hoped that $\mathbf{F}_q$ is a finite union of well-ordered chains. Clearly, this is so if and only if there exist $\mathbf{P}_1, \ldots, \mathbf{P}_k \in \mathbf{F}_q$ such that $\mathbf{I}_q = \mathbf{P}_1 \cap \cdots \cap \mathbf{P}_k$. Again, slightly reformulating this hope, we are led to the following question [3, Question 15].

Question. Let Θ be a homomorphism from $\mathbf{C}(X_0)$ into a Banach algebra. Do there exist prime ideals $\mathbf{P}_1, \ldots, \mathbf{P}_k$ in $\mathbf{C}(X_0)$ such that $\Theta \mid \mathbf{P}_1 \cap \cdots \cap \mathbf{P}_k$ is continuous?

The strongest form of the theorem asserting the existence of discontinuous homomorphisms from $\mathbf{C}(X_0)$ is the following [9]. We denote the cardinality of a set S by $|S|$. Let $\mathbf{P}$ be a nonmaximal, prime ideal of $\mathbf{C}(X_0)$ such that $|\mathbf{C}(X_0)/\mathbf{P}| = \aleph_1$. Then there exists a homomorphism from $\mathbf{C}(X_0)$ into a Banach algebra such that $\ker \Theta = \mathbf{P}$. If X_0 is infinite, then $\mathbf{C}(X_0)$ contains nonmaximal, prime ideals $\mathbf{P}$ such that $|\mathbf{C}(X_0)/\mathbf{P}| = 2^{\aleph_0}$, and so, if CH holds, there are discontinuous homomorphisms from $\mathbf{C}(X_0)$. Further, if $|\mathbf{C}(X_0)| = 2^{\aleph_0}$ and if CH holds, then each ideal of $\mathbf{C}(X_0)$ which is a finite intersection of nonmaximal, prime ideals is the kernel of such a homomorphism. Thus, a positive answer to the question would give the complete story about the domains of continuity of homomorphisms from $\mathbf{C}(X_0)$, at least if $|\mathbf{C}(X_0)| = 2^{\aleph_0}$. (Incidently, we do not know whether or not every infinite-dimensional, nonnilpotent commutative Banach algebra $\mathbf{A}$ contains a nonmaximal, prime ideal $\mathbf{P}$ such that $|\mathbf{A}/\mathbf{P}| = 2^{\aleph_0}$. If $\mathbf{A}$ is separable and an integral domain, then $\{0\}$ is certainly such an ideal, and it may be the only one [11].)

In attempting to answer our question, we reformulate it again by using a result of Johnson [12]. First, some notation that will be fixed for the remainder of this note. Let X be a locally compact space, let $\mathbf{C}_0(X)$ be the space of continuous functions on X which vanish at infinity, and denote by βX the Stone-Čech compactification of X. For $f \in \mathbf{C}_0(X)$, let $\hat{f}$ be the continuous extension of f to βX. For $p \in \beta X \setminus X$, set

$$\mathbf{J}_p = \left\{ f \in \mathbf{C}_0(X) \colon \hat{f}^{-1}(\{0\}) \text{ is a neighborhood of } p \text{ in } \beta X \right\}.$$

Johnson's theorem, which extends that of Bade and Curtis, is the following. Let X_0 be a compact space, let $\Theta \colon \mathbf{C}(X_0) \to \mathbf{B}$ be a discontinuous homomorphism with singularity set F, and let $X = X_0 \setminus F$. Identify $\mathbf{C}_0(X)$ with $\{ f \in \mathbf{C}(X_0) \colon f \mid F = 0\}$. Then there exist $p_1, \ldots, p_n \in \beta X \setminus X$, a continuous homomorphism $\mu \colon \mathbf{C}(X_0) \to \mathbf{B}$, and linear maps $\nu_1, \ldots, \nu_n \colon \mathbf{C}(X_0) \to \mathbf{B}$ such that $\Theta = \mu + \nu_1 + \cdots + \nu_n$ and $\nu_i \mid \mathbf{C}_0(X)$ is a homomorphism with $\nu_i \mid \mathbf{J}_p = 0$ $(i = 1, \ldots, n)$. Necessarily, the range of each ν_i is contained in the radical of $\mathbf{B}$. A direct proof of this result is given in [4, Chapter 1].

Thus the question we raised is equivalent to the following. Let X be a locally compact space, let $p \in \beta X \setminus X$, and let $\nu \colon \mathbf{C}_0(X) \to \mathbf{R}$ be a homomorphism from $\mathbf{C}_0(X)$ into a radical Banach algebra $\mathbf{R}$ such that $\ker \nu \supset \mathbf{J}_p$. Is $\ker \nu$ a finite intersection of prime ideals?

The purpose of the present note is to make a modest start towards this problem by analysing when we can have $\ker \nu$ equal to $\mathbf{J}_p$ in the above situation, for suitably restricted X. We also determine when $\mathbf{J}_p$ is itself a prime ideal.

Results. Let X be a locally compact space. For $S \subset X$, denote by $\bar{S}$ the closure of S in βX. If U is open in βX, then $\overline{U \cap X} = \bar{U}$. If S and T are zero-sets of continuous functions on X, then $\bar{S} \cap \bar{T} = \overline{S \cap T}$; in particular, if X is metrizable, this holds for any closed sets S and T in X. See [**10**, Chapter 6] for the elementary properties of βX. Denote by W the (open) subset of X consisting of isolated points, and by K the (closed) subset of X consisting of nonisolated points. Define

$$P = (\overline{W} \setminus \overline{K}) \cup \{ p \in \overline{K} \setminus \overline{W}\colon p \text{ is not a limit point in } \beta X \text{ of a discrete subset of } K \setminus \overline{W} \}.$$

THEOREM. *Let X be a locally compact, σ-compact, metrizable space, and suppose that CH holds. Then for a point $p \in \beta X \setminus X$ the following are equivalent*:

(1) $p \in P$;

(2) $\mathbf{J}_p$ *is a prime ideal*;

(3) $\mathbf{J}_p$ *is a finite intersection of prime ideals*;

(4) $\mathbf{J}_p$ *is the kernel of a discontinuous homomorphism from* $\mathbf{C}_0(X)$ *into a* (*radical*) *Banach algebra.*

PROOF. Since $\beta X = \overline{K} \cup \overline{W}$, the point $p \in \beta X$ lies in exactly one of $\overline{K} \cap \overline{W}$, $\overline{W} \setminus \overline{K}$, and $\overline{K} \setminus \overline{W}$; we consider the three possibilities in turn. Let d denote the metric on X, and, for $x \in X$ and $\varepsilon > 0$, let $B(x, \varepsilon) = \{ y \in X\colon d(x, y) < \varepsilon\}$.

Suppose first that $p \in \overline{K} \cap \overline{W}$, so that $p \notin P$. Note that

$$\overline{K} \cap \overline{W} = (K \cap \overline{W} \cap X)^{-} = (K \cap \overline{W})^{-}$$

since both K and $\overline{W} \cap X$ are closed in X. Since X is σ-compact, we can write $K \cap \overline{W} = \bigcup E_n$, where each E_n is a compact subset of X, and $E_n \subset E_{n+1}$ (strictly) $(n = 1, 2, \ldots)$. Noting that $\bigcup E_n \subset \overline{W} \setminus W$, inductively construct finite subsets W_n of W, and a sequence (α_n) of positive reals as follows. Choose $x \in W$, set $\alpha_1 = 1$, $W_1 = \{x\}$. Given $W_1, \ldots, W_n$, set $\delta_n = d(E_{n+1}, W_1 \cup \cdots \cup W_n)$, and take $0 < \alpha_{n+1} < \min(\frac{1}{2}\delta_n, \alpha_n, n^{-1})$. Take a finite cover of E_{n+1} by open balls of radius α_{n+1}, and choose one point of W in each ball to form the set W_{n+1}. By the construction $W_i \cap W_j = \varnothing$ if $i \neq j$.

Let $\{\sigma_m\colon m \in \mathbf{N}\}$ be an infinite partition of $\mathbf{N}$ into infinite sets, and let $A_m = \bigcup\{W_n\colon n \in \sigma_m\}$. Let $f_n = 1/n$ on the set W_n and 0 elsewhere. Since W_n is a finite set of isolated points, $f_n \in \mathbf{C}_0(X)$. Set $g_m = \Sigma\{f_n\colon n \in \sigma_m\}$, so that $g_m \in \mathbf{C}_0(X)$ $(m = 1, 2, \ldots)$.

Now take a neighborhood U of p in βX. Since $p \in (K \cap \overline{W})^{-}$, there is a point $x \in K \cap \overline{W} \cap U$; take $\varepsilon > 0$ so that $B(x, \varepsilon) \subset U$ and choose k such that $\alpha_k < \frac{1}{2}\varepsilon$. Since σ_m is infinite, there is an $n \in \sigma_m$ with $n > k$. But then $W_n \cap B(x, \varepsilon) \neq \varnothing$, hence $A_m \cap U \neq \varnothing$, and so $g_m \notin \mathbf{J}_p$. On the other hand, $g_m g_n = 0$ for $m \neq n$ because $A_m \cap A_n = \varnothing$, and so [**1**, Theorem 5.3] shows that $\mathbf{J}_p$ cannot be the kernel of a radical homomorphism.

Secondly, consider the case where $p \in \overline{W} \setminus \overline{K}$, so that $p \in P$. Take $f, g \in \mathbf{C}_0(X)$ with $fg \in \mathbf{J}_p$. Let U be a neighborhood of p in βX with $\overline{U} \cap \overline{K} = \varnothing$, and set

$$V_1 = \{ x \in \overline{U} \cap X\colon |f(x)| < |g(x)| \},$$

$$V_2 = \{ x \in \overline{U} \cap X\colon |f(x)| \geqslant |g(x)| \}.$$

Since V_1, $V_2 \subset W$, both these sets are open in X. But $V_1 \cup V_2 = \overline{U} \cup X$, which is closed in X. Thus both V_1 and V_2 are clopen in X. Hence $\overline{V}_1 \cap \overline{V}_2 = \overline{V_1 \cap V_2} = \varnothing$, and so exactly one of $\overline{V}_1$ and $\overline{V}_2$ is a neighborhood of p. Since on a possibly smaller neighborhood fg vanishes, it follows that one of f and g lies in $\mathbf{J}_p$, and hence that $\mathbf{J}_p$ is prime.

Thirdly, suppose that $p \in \overline{K} \setminus \overline{W}$, and take the case that p is a limit point of a discrete subset D of $K \setminus \overline{W}$, so that $p \notin P$. Since X is second countable, D must be countable, say $D = \{x_n\}$. Let $\{U_n\}$ be a collection of open sets in X with $x_n \in U_n$ and $U_n \cap U_m = \varnothing$ $(n, m \in \mathbf{N}, n \neq m)$. Further, since each x_n is nonisolated in X, there exist $\{x_m^n\} \subset U_n \setminus \{x_n\}$ with $x_m^n \to x_n$ as $m \to \infty$, and $x_r^n \neq x_s^n$ if $r \neq s$. Now choose open sets $V_m^n \subset U_n$ with $x_m^n \in V_m^n$ and $V_r^n \cap V_s^n = \varnothing$ if $r \neq s$. Finally, choose continuous, nonnegative real-valued functions $\{f_n\}$ on X_0 such that

(a) $\operatorname{supp}(f_m) \subset \bigcup\{V_m^n\colon n \in \mathbf{N}\}$;

(b) $f_m(x_m^n) = (n+m)^{-1} = \sup\{f_m(x)\colon x \in V_m^n\}$.

Let $\{\sigma_m\colon m \in \mathbf{N}\}$ be an infinite partition of $\mathbf{N}$ into infinite sets and let

$$F_m = \sum\{f_n\colon n \in \sigma_m\}.$$

The F_m are well defined since the summands have disjoint support, and lie in $\mathbf{C}_0(X)$ by (b). If Z is a neighborhood of p in βX, then $x_n \in Z$ for some n, whence for each i, $x_j^n \in Z$ for $j \in \sigma_i$ with j sufficiently large. Thus F_i is not identically zero on $Z \cap X$, that is, $F_i \notin \mathbf{J}_p$. But $F_iF_j = 0$ if $i \neq j$, and so, as above, $\mathbf{J}_p$ cannot be the kernel of a discontinuous homomorphism.

There remains the case where $p \in \overline{K} \setminus \overline{W}$, and p is not a limit point of a discrete subset of $K \setminus \overline{W}$. Here we show that $\mathbf{J}_p$ is again prime. Suppose then that f and $g \in \mathbf{C}_0(X)$ with $fg \in \mathbf{J}_p$. Let

$$G_1 = \{x \in X\colon |f(x)| \leqslant |g(x)|\},$$
$$G_2 = \{x \in X\colon |f(x)| \geqslant |g(x)|\},$$
$$G = \{x \in X\colon |f(x)| = |g(x)|\},$$

so that $G = G_1 \cap G_2$, As above, if $p \notin \overline{G}_1$ or $p \notin \overline{G}_2$, then one of f or g lies in $\mathbf{J}_p$. Thus we may suppose that $p \in \overline{G}$, that is that $p \in (\overline{G} \cap \overline{K}) \setminus \overline{W} = \overline{G \cap K \setminus W}$.

Let $Y = \operatorname{int}_K(G \cap K)$ and $Z = \partial_K(G \cap K)$, the frontier of $G \cap K$ in K. Since $Z = \partial_K R$, where $R = \{x \in K\colon |f(x)| \neq |g(x)|\}$, an open set in K, it follows from a theorem of Hausdorff [**14**, Lemma 4.39] that Z is the set of limit points (in K) of a discrete subset D of K. But then $\overline{Z} \subset \overline{D}$, and so by hypothesis $p \notin \overline{Z}$. Let U be a neighborhood of p in βX such that $\overline{U} \cap (\overline{Z} \cup \overline{W}) = \varnothing$ and $\hat{f}\hat{g}\,|\,U = 0$, and set $K_1 = \overline{U} \cap G$, $K_2 = \overline{U} \cap (K \setminus Y)$. Then K_1 and K_2 are closed in K, and hence closed in X. Also, $K_1 \cap K_2 = \varnothing$ and $K_1 \cup K_2 = \overline{U} \cap X$, so that $\overline{K}_1 \cup \overline{K}_2 = \overline{U}$ and $\overline{K}_1 \cap \overline{K}_2 = \varnothing$. Since $p \in \overline{G \cap K} \subset \overline{K}_1$, it follows that $\overline{K}_1$ is a neighborhood of p. But $|f| = |g|$ on G, and hence on $\overline{K}_1$, and so $f = g = 0$ on $\overline{K}_1$. Thus f and g belong to $\mathbf{J}_p$.

We have now proved that (1) $\Rightarrow$ (2) and that (4) $\Rightarrow$ (1); (2) $\Rightarrow$ (3) is trivial. For (3) $\Rightarrow$ (4), note that X is metrizable, separable and infinite, so that $\mathbf{C}_0(X)$ is infinite dimensional and separable. In particular, $|\mathbf{C}_0(X)| = 2^{\aleph_0}$, and so by [**9**, Théorème

6.2] each nonclosed prime ideal of $\mathbf{C}_0(X)$ is the kernel of a discontinuous homomorphism; it is immediate that the same is true for a finite intersection of prime ideals. □

Remarks. 1. The topological hypotheses on X include the case where $X = X_0 \setminus F$, X_0 compact and metrizable, F a finite set of nonisolated points.

2. The metrizability of X is heavily used in the proof, both for the details of the construction, and to ensure that arbitrary closed sets are zero-sets, though where possible the weaker supposition of second countability is explicitly used.

3. Recall that a point $p \in \beta X$ is called *remote* [**14**, 4.37] if $p \notin \overline{D}$ for any discrete set $D \subset X$. Thus if the set W of isolated points has compact closure in X, then condition (1) of the theorem is just the statement that p is remote. Note that with this condition on W and assuming CH there are $2^{\mathfrak{c}}$ remote points [**14**, Theorem 4.43].

Our argument shows in fact that, if p is remote, then $\mathbf{J}_p$ is prime, but the converse to this is false. For let $X_0 = \mathbf{N} \cup \{\infty\}$, and $X = \mathbf{N}$. Then each $\mathbf{J}_p$ is prime, but no $p \in \beta X \setminus X$ can be remote. Note that the result of [**1**] shows that in this case any homomorphism on $\mathbf{C}(X_0)$ is continuous on some finite intersection of the prime ideals $\{\mathbf{J}_p\colon\ p \in \beta X \setminus X\}$.

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