A $JC$-ALGEBRA WHICH IS NOT THE RANGE OF A POSITIVE PROJECTION ON A $C^*$-ALGEBRA

A. GUYAN ROBERTSON

ABSTRACT. An example is given of a $JC$-algebra which is not the range of any positive projection on the selfadjoint part of its generated $C^*$-algebra.

$JC$-algebras arise naturally in the study of positive projections on operator algebras [1, Theorem 1.4]. Effros and Størmer showed in [1, Theorem 2.1] that if $A$ is a simple $JC$-algebra and $B$ is the $C^*$-algebra generated by $A$, then there exists a positive projection $P$ on $B$ such that $P(B_{sa}) = A$. Furthermore they expressed the belief that this result may fail for general $JC$-algebras [1, Remark 2.7]. The purpose of this note is to provide an explicit example showing that this is indeed the case.

The example is constructed as follows. Let $F$ be the CAR-algebra, acting irreducibly on a Hilbert space $H$. By [2, Theorem 6.2.2], $F$ is the $C^*$-algebra generated by an infinite-dimensional spin factor $V$. If $K$ denotes the algebra of compact operators on $H$, then $F \cap K = \{0\}$ [3, Theorem 6.5.7]. Let $A = K_{sa} + V$ and let $B$ be the $C^*$-algebra generated by $A$. Then $A$ is a $JC$-algebra and $B = K + F$. These statements follow easily from [3, Corollary 1.5.8] and its Jordan-algebraic analogue [2, 3.4].

THEOREM. There is no positive projection $P$ on $B$ such that $P(B_{sa}) = A$.

PROOF. Suppose that such a $P$ exists. We show that this implies that $V$ is reversible in the sense that it is closed under symmetric products of the form $a_1a_2 \cdots a_n + a_n \cdots a_2a_1$ [2, 2.3.2]. This will contradict [2, Theorem 6.2.5], which asserts that $V$ is not reversible.

Given $a_1, \ldots, a_n \in V$, let $a = a_1a_2 \cdots a_n + a_n \cdots a_2a_1$. If $b \in K_{sa}$ then $a \circ b = \frac{1}{2}(ab + ba) \in K_{sa}$. Therefore $a \circ b = P(a \circ b) = P(a) \circ b$, where the last equality follows from [1, Lemma 1.1] together with the fact that the range of $P$ is assumed to be a Jordan algebra. Now the identity operator is a strong limit of elements of $K_{sa}$. Hence $a = P(a) \in A$. It follows that $a = k + x$, where $k \in K_{sa}$ and $x \in V$. Since it is obvious from its definition that $a \in F$, we have

$$a - x = k \in F \cap K = \{0\}.$$ 

Therefore $a = x \in V$. This shows that $V$ is reversible, contradicting [2, Theorem 6.2.5].

REMARK. The argument above actually shows that $A$ is not the range of a positive projection on any reversible $JC$-algebra containing $A$ as a $JC$-subalgebra.
REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF EDINBURGH, EDINBURGH EH9 3JZ, SCOTLAND