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ON THE VOLUMES OF IMAGES OF HOLOMORPHIC MAPPINGS IN $\mathbf{C}^n$

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ABSTRACT. Let f be a holomorphic map from the unit ball in complex n-space to complex n-space. We establish a lower bound for the volume, *taken without multiplicity*, of the image of f. The estimate is in terms of the boundary values of f. This generalizes some known results in one complex variable. The proof uses the methods of uniform algebras.

1. Introduction. We begin by quoting a result on uniform algebras [**1**].

THEOREM A. *Let X be a compact Hausdorff space which is the maximal ideal space of a uniform algebra $\mathcal{A}$ on X. Let $x_0 \in X$ and let μ be a representing measure on X for the homomorphism on $\mathcal{A}$ of evaluation at x_0. Then*

$$\pi \int |f|^2 \, d\mu \leq \operatorname{area}(f(X))$$

for all $f \in \mathcal{A}$ such that $f(x_0) = 0$.

Here the area and, more generally, the volumes which appear below are taken as volume measures of sets, without regard to multiplicity. An interesting application of this has been given recently by Axler and Shapiro [**3**]. Applying Theorem A to the disk algebra yields the following, which was first obtained by Alexander, Taylor and Ullman [**2**]. Here U denotes the open unit disk.

THEOREM B. *Let f be a holomorphic function on U with $f(0) = 0$. Suppose that f belongs to the Nevanlinna class and (so) has boundary values f^* a.e. on the unit circle. Then*

$$\pi \int |f^*(e^{i\theta})|^2 \frac{d\theta}{2\pi} \leq \operatorname{area}(f(U)).$$

We shall generalize this to several variables. Fix the dimension $n \geq 1$ and let B be the open unit ball in $\mathbf{C}^n$ with c_n its $2n$-dimensional volume.

THEOREM 1. *Let f be a holomorphic mapping on $\overline{B}$ to $\mathbf{C}^n$ with $f(0) = 0$. There exists a representing measure μ on ∂B for evaluation at the origin for the ball algebra such that*

$$c_n \int |f|^{2n} \, d\mu \leq 2n\text{-volume}(f(B)). \tag{1.1}$$

REMARKS. (i) For $f = (f_1, f_2, \ldots, f_n)$ we write $|f(z)|^2 = \sum |f_k(z)|^2$.

(ii) For $n = 1$, $d\theta/2\pi$ is the unique representing measure for the ball (= disk) algebra and so we recover Theorem B.

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(iii) Note that the integrand $|f|^{2n}$ does not involve derivatives of f. However, the representing measure μ does depend upon f and, if written out, would involve its derivatives.

Our second result is a variant of Theorem 1 not involving a general representing measure but instead the solution to a Bremermann-Dirichlet problem [**5**].

THEOREM 2. *Let f be as in Theorem 1. Let u be the solution to the Bremermann-Dirichlet problem in the unit ball with boundary values $|f|^{2n}$. Then*

$$c_n u(0) \leq 2n\text{-volume}(f(B)). \tag{1.2}$$

REMARKS. (i) For $n = 1$, u is just the harmonic extension to the disk of $|f|^2$ on the unit circle and so $u(0) = \int |f(e^{i\theta})|^2 \, d\theta/2\pi$. Thus we again recover Theorem B.

(ii) Although we have stated Theorems 1 and 2 in the equidimensional case as it seems to be of most interest, they remain valid for mappings $\mathbf{C}^m \to \mathbf{C}^n$.

Theorems 1 and 2 give lower bounds for $\text{vol}(f(B))$. The next result addresses the question of when these bounds are trivial. Here $Z(f) = \{f = 0\} = \{z \in B\colon f_k(z) = 0, 1 \leq k \leq n\}$ and u is as in Theorem 2.

THEOREM 3. *The following are equivalent.*

(i) $u(0) = 0$.

(ii) *0 is not an isolated point of $Z(f)$; i.e., the local dimension of $Z(f)$ at the origin is positive.*

(iii) *There exists a representing measure μ on ∂B for evaluation at the origin for the ball algebra such that*

$$\int |f|^p \, d\mu = 0 \tag{1.3}$$

for some $p > 0$.

Our final result is a generalization of the following fact about the Green's function [**4**, p. 61].

THEOREM C. *Let Ω be a domain in $\mathbf{C}$ containing the origin and let $G(z)$ be the Green's function with pole at the origin. Then*

$$2 \int_\Omega G(z) \, dx \, dy \leq \text{area}(\Omega). \tag{1.4}$$

Ch. Stanton [**8**] has recently given a proof of Theorem A based in part on this formula. On the other hand, (1.4) is in turn a consequence of Theorem A. To see this, first observe that, by approximating Ω from within, we can assume that $\partial\Omega$ is smooth. Let μ be harmonic measure for 0; μ is supported on $\partial\Omega$ and can be expressed as the differential $1/2\pi * dG$. An application of Green's formula (cf. Lemma 4 below) yields

$$\frac{2}{\pi} \int_\Omega G \, dx \, dy = \int_{2\pi} |z|^2 \, d\mu. \tag{1.5}$$

Now μ is a representing measure for evaluation at the origin for the algebra $R(\overline{\Omega})$ with maximal ideal space $\overline{\Omega}$. Applying Theorem A to the identity function $z \in R(\overline{\Omega})$ gives

$$\pi \int |z|^2 \, d\mu \leq \text{area}(\overline{\Omega}). \tag{1.6}$$

The inequality (1.4) follows from (1.5) and (1.6).

For a generalization to several variables let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$ containing the origin. Let v be the solution to the Bremermann-Dirichlet problem on Ω with boundary values $\log|z|$. Set $G(z) = v - \log|z|$ in Ω. Then G is a natural generalization of the Green's function with logarithmic pole at the origin; $G \geq 0$ is not psh but $G + \log|z|$ is psh. Let dV be Lebesgue measure in $\mathbf{C}^n$. We have

THEOREM 4.

$$2n \int_\Omega G\, dV \leq 2n\text{-volume}(\Omega). \tag{1.7}$$

2. The proofs. Let σ be Haar measure on ∂B. For a compact subset X of $\mathbf{C}^n$ and $\alpha \in \partial B$ we write

$$L_X(\alpha) = \{t \in \mathbf{R}\colon\ t > 0 \text{ and } t\alpha \in X\}$$

and

$$A_X(\alpha) = \{re^{i\theta} \in \mathbf{C}\colon\ r > 0,\ \theta \text{ real, and } re^{i\theta}\alpha \in X\}.$$

LEMMA 1 (COMPLEX POLAR COORDINATES).

$$2n\text{-vol}(X) = a_n \int_{\partial B} \left(\int_{A_X(\alpha)} r^{2n-1}\, dr\, d\theta \right) d\sigma(\alpha) \tag{2.1}$$

where (r, θ) are polar coordinates on $A_X(\alpha)$ and a_n is a constant.

PROOF. The integral in (2.1) equals

$$\int_{\partial B} \left\{ \int_0^{2\pi} \left(\int_{L_X(e^{i\theta}\alpha)} r^{2n-1}\, dr \right) d\theta \right\} d\sigma(\alpha)$$
$$= \int_{\partial B} \left\{ \int_0^{2\pi} \left(\int_{L_X(\alpha)} r^{2n-1}\, dr \right) d\theta \right\} d\sigma(e^{i\theta}\alpha)$$

where we have used the substitution $\alpha \to e^{-i\theta}\alpha$. By the invariance of σ this becomes

$$2\pi \int_{\partial B} \left(\int_{L_X(\alpha)} r^{2n-1}\, dr \right) d\sigma(\alpha).$$

By the usual formula for real polar coordinates this last integral is, up to a constant factor, the $2n$-dimensional volume of X.

LEMMA 2 [**1**, THEOREM 3.1]. *Let X, $\mathcal{A}$, x_0, and μ be as in Theorem* A. *Let $g \in \mathcal{A}$ with $g(x_0) = 0$. Then for $r \geq 0$*

$$2\pi\mu\{x\colon\ |g(x)| \geq r\} \leq \Theta\{e^{i\theta}\colon\ re^{i\theta} \in g(X)\} \tag{2.2}$$

where Θ denotes angular measure; i.e., linear measure on the unit circle.

LEMMA 3. *In the notation of the previous lemma,*

$$\int |g|^{2n}\, d\mu \leq \frac{n}{\pi} \int_{g(X)} r^{2n-1}\, dr\, d\theta \tag{2.3}$$

where $n \geq 1$ is an integer and (r, θ) are polar coordinates on $\mathbf{C}$.

PROOF. Let $M(r) = \mu\{x\colon |g(x)| \geq r\}$. By a standard result [**6**, p. 42] in integration theory we have

$$\int_0^\infty 2nr^{2n-1}M(r)\,dr = \int |g|^{2n}\,d\mu. \tag{2.4}$$

Now multiply (2.2) by nr^{2n-1}, integrate over r from 0 to ∞ and apply (2.4) to get (2.3).

For a compact set Y of $\mathbf{C}^n$ we let $P(Y)$ ($R(Y)$ resp.) denote the uniform algebra on Y which is the uniform closure of the polynomials (rational function with poles off Y, resp.). The polynomially convex hull of Y, denoted $\hat{Y}$, is identified with the maximal ideal space of $P(Y)$. In particular, if V is an analytic subvariety of B without isolated points which extends to be analytic on $\overline{B}$, then $\overline{V}$ is polynomially convex and the Shilov boundary of $P(\overline{V})$ is $\overline{V}\cap\partial B$, by the maximum principle. We shall apply Lemma 3 to $P(\overline{V})$ below. Let Jf be the complex Jacobian determinant of f. For the proofs of Theorems 1 and 2 we initially assume $Jf(0) \neq 0$.

PROOF OF THEOREM 1. Fix $\alpha \in \partial B$ and let l_α be the complex line through α. Let V_α be the analytic subvariety of B given by $B\cap f^{-1}(l_\alpha)$. By our assumption that $Jf(0) \neq 0$, $0 \in V_\alpha$ is not an isolated point. Hence there is a representing measure μ_α on $\partial V_\alpha \subseteq \partial B$ for evaluation at the origin for the algebra $P(\overline{V}_\alpha)$. There exists an analytic function g on $\overline{V}_\alpha$ such that $f(x) = g(x)\cdot\alpha$ for all $x \in V_\alpha$; namely, $g = \sum \overline{\alpha}_k f_k$. Applying Lemma 3 to $g \in P(\overline{V}_\alpha)$ we get

$$\int |g|^{2n}\,d\mu_\alpha \leq \frac{n}{\pi}\int_{g(\overline{V}_\alpha)} r^{2n-1}\,dr\,d\theta. \tag{2.5}$$

Since $|g| = |f|$ and $g(\overline{V}_\alpha) = A_{f(\overline{B})}(\alpha)$ we have

$$\int |f|^{2n}\,d\mu_\alpha \leq \frac{n}{\pi}\int_{A_{f(\overline{B})}(\alpha)} r^{2n-1}\,dr\,d\theta. \tag{2.6}$$

Now integrate (2.6) over $\alpha \in \partial B$. On the right-hand side we get, by Lemma 1, $c_n \cdot 2n\text{-vol}(f(\overline{B}))$, for some constant c_n. On the left-hand side we obtain $\int |f|^{2n}\,d\mu$ where μ is the measure on ∂B induced by the functional

$$h \mapsto \int_{\partial B}\left(\int h\,d\mu_\alpha\right)d\sigma(\alpha)$$

for $h \in \mathcal{C}(\partial B)$. As an average of representing measures, μ is itself a representing measure. (It should be noted that the μ_α's can be chosen so that $\alpha \mapsto \mu_\alpha$ is weak-* measurable; i.e., $\alpha \mapsto \int h\,d\mu_\alpha$ is measurable on ∂B for all $h \in \mathcal{C}(\partial B)$—we omit the details.)

This gives the theorem except for the determination of the constant c_n. For this, observe that all of the inequalities of the proof become equalities when f is the identity map. Hence in this case equality holds in the theorem and says $c_n = 2n\text{-vol}(B)$.

PROOF OF THEOREM 2. We shall use the notation of the proof of Theorem 1. Fix $\alpha \in \partial B$. Let W_α be the pure one-dimensional component of V_α such that $0 \in W_\alpha$. Let μ_α be harmonic measure on $\partial W_\alpha \subseteq \partial B$ for the origin. Let h be

the solution to the Dirichlet problem on W_α with boundary values $|f|^{2n}$. Then by definition of harmonic measure we have

$$h(0) = \int |f|^{2n}\, d\mu_\alpha. \tag{2.7}$$

Since u is psh, its restriction to W_α is subharmonic. Since $u = h$ on ∂W_α and since h is harmonic on W_α we get

$$u(0) \le h(0). \tag{2.8}$$

Hence we obtain, using (2.6)–(2.8),

$$u(0) \le \frac{n}{\pi} \int_{A_{f(\overline{B})}(\alpha)} r^{2n-1}\, dr\, d\theta.$$

Integrating with respect to $d\sigma(\alpha)$ gives the theorem.

We now complete the proofs of Theorems 1 and 2 when $Jf(0) = 0$. There are two cases to consider:

(a) $Jf \not\equiv 0$. Choose $x_n \to 0$ in B such that $Jf(x_n) \neq 0$. Let $\{\varphi_n\}$ be Möbius transformations of the ball such that $\varphi_n(0) = x_n$ and $\varphi_n \to$ identity. Set $f_n = f \circ \varphi_n - f(x_n)$; $Jf_n(0) \neq 0$, $f_n(0) = 0$, and $f_n \to f$ uniformly on $\overline{B}$. Then Theorems 1 and 2 for f_n yield a representing measure μ_n and a Bremermann-Dirichlet function u_n. Clearly u_n converges to u uniformly on $\overline{B}$; let μ be a weak-* limit of $\{\mu_n\}$. It is straightforward to check that the validity of Theorems 1 and 2 for f_n continues to hold also for f in the limit.

(b) $Jf \equiv 0$. Then $f^{-1}(0) = V$ has positive dimension. Let μ be a representing measure on $\partial V \subseteq \partial B$ for the origin. Then as $f \equiv 0$ on ∂V we have $\int |f|^{2n}\, d\mu = 0$. In this case both sides of (1.1) reduce to zero. Similarly, by the maximum principle, $u(0) \le \max_{\partial V} u = 0$ and so both sides of (1.2) are also zero.

PROOF OF THEOREM 3. (i) $\Rightarrow$ (ii). We argue by contradiction. Suppose that 0 is an isolated point of $Z(f)$. Choose $0 < \delta < 1$ and $\varepsilon > 0$ such that $|f| \ge \varepsilon$ on ∂B_δ where $B_\delta = \{z\colon \|z\| < \delta\}$. Let u_δ be the solution of the Bremermann-Dirichlet problem in B_δ with boundary values $|f|^{2n}$. Set, for $z \in B$,

$$u'(z) = \begin{cases} u_\delta(z), & \|z\| < \delta, \\ |f(z)|^{2n}, & \delta \le \|z\| < 1. \end{cases}$$

Then u' is a "modification" of the psh function $|f|^{2n}$ and so is itself psh. Since $\varepsilon \le u_\delta$ on B_δ, it follows in particular that $\varepsilon \le u_\delta(0)$. As u' has boundary values $\le |f|^{2n}$ we have $u' \le u$ on B. Thus we get

$$0 < \varepsilon \le u_\delta(0) = u'(0) \le u(0).$$

This contradicts (i).

(ii) $\Rightarrow$ (iii). Let V be an analytic component of $Z(f)$ through the origin. By (ii), V has positive dimension and so $\partial V \subseteq \partial B$. By the maximum principle, there is a representing measure μ on ∂V for the origin. Since $f \equiv 0$ on ∂V we get $\int |f|^p\, d\mu = 0$ for any $p > 0$.

(iii) $\Rightarrow$ (i). Let $\Gamma \subseteq \partial B$ be the support of μ. Then (1.3) implies that $|f| \equiv 0$ on Γ and so $u \equiv 0$ on Γ. Since μ represents 0 on Γ, we have $0 \in \hat{\Gamma}$. Hence, since u is psh, it follows [**7**] that

$$u(0) \le \sup_{\Gamma} u = 0.$$

As u is nonnegative, we conclude that $u(0) = 0$.

LEMMA 4. *Let W be a domain with smooth boundary in $\mathbf{C}$, containing the origin and let $g(z)$ be the Green's function of W with pole at the origin. Let μ be harmonic measure on ∂W for the origin. Then for $p \geq 1$*

$$\frac{2p^2}{\pi}\int_W |z|^{2(p-1)} g(z)\,dx\,dy = \int |z|^{2p}\,d\mu. \tag{2.9}$$

PROOF. Harmonic measure μ can be expressed as $1/2\pi * dg$ on ∂W. Applying Green's formula to

$$\int |z|^{2p}\,d\mu = \frac{1}{2\pi}\int_{\partial W} |z|^{2p} * dg$$

yields (2.9).

PROOF OF THEOREM 4. Let $\alpha \in \partial B$ and let W_α be the intersection of Ω with the complex line through α. Assume that ∂W_α is smooth—this is true for almost all α. Let μ_α be harmonic measure on ∂W_α for the origin relative to W_α and let g_α be the Green's function for W_α with pole at the origin. By Lemma 4 we have

$$\frac{2n^2}{\pi}\int_{A_{\overline{\Omega}}(\alpha)} |z|^{2(n-1)} g_\alpha(z)\,dx\,dy = \int |z|^{2n}\,d\mu_\alpha \tag{2.10}$$

where we view $\overline{W}_\alpha$ as the plane domain $A_{\overline{\Omega}}(\alpha)$.

The restriction of $G + \log|z|$ to W_α is subharmonic while $g_\alpha + \log|z|$ is harmonic on W_α. Also $G + \log|z| = \log|z| = g_\alpha + \log|z|$ on ∂W_α. Hence we have $G \leq g_\alpha$ on W_α. The argument leading to (2.6) shows that

$$\int |z|^{2n}\,d\mu_\alpha \leq \frac{n}{\pi}\int_{A_{\overline{\Omega}}(\alpha)} r^{2n-1}\,dr\,d\theta.$$

Hence from (2.10) we get

$$2n\int_{A_{\overline{\Omega}}(\alpha)} G(z) r^{2n-1}\,dr\,d\theta \leq \int_{A_{\overline{\Omega}}(\alpha)} r^{2n-1}\,dr\,d\theta. \tag{2.11}$$

Now integrating (2.11) with respect to $d\sigma(\alpha)$ and applying Lemma 1 yields (1.7).

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