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SIMPLE EXAMPLES OF NONREALIZABLE CR HYPERSURFACES

HOWARD JACOBOWITZ

ABSTRACT. A new proof is provided of a nonrealizability result due to Hill, Penrose, and Sparling. This result is then generalized to higher dimensions: Each $\overline{\partial}_b$-cohomology class in $H^{0,1}(M)$ can be used to define a nonrealizable CR structure on $M \times \mathbf{R}^2$.

Consider an example due to Hill, Penrose, and Sparling [**P**] as formulated by Eastwood [**E**]. Let M be a three-dimensional hypersurface in $\mathbf{C}^2$ with Lewy operator L. Define a CR structure on $M \times \mathbf{C}$ by taking

$$L_1 = L + g\varsigma\partial/\partial\varsigma, \qquad L_2 = \partial/\partial\overline{\varsigma}, \tag{1}$$

where g is a function on M and ς is the coordinate for $\mathbf{C}$. It is known that when $Lf = g$ has no solution, then this structure cannot be realized as a hypersurface in $\mathbf{C}^3$. The proof outlined in [**P**] uses the extension of holomorphic vector bundles across boundaries in the base. A simple proof using a Taylor series expansion was given in [**E**]. Here we give another simple proof; this one uses the canonical bundle K of a CR manifold. This bundle has also been useful in other contexts [**F**, **J**]. If M^{2n+1} is a CR manifold of hypersurface type and if $\theta_1, \theta_2, \ldots, \theta_{n+1}$ are independent forms each of which annihilates every Lewy vector field, then $K = \{\lambda\theta_1 \wedge \cdots \wedge \theta_{n+1}, \lambda \in \mathbf{C}^*\}$. Note that K does not depend on the particular choices of θ_j.

So now let $\theta_1 = dz_1$ and $\theta_2 = dz_2$ where (z_1, z_2) are the coordinates on $\mathbf{C}^2$ restricted to M. We may assume $dz_1 d\overline{z}_1 dz_2 \neq 0$ and also that $Lz_1 = 0$, $Lz_2 = 0$, $L\overline{z}_1 = 1$. Let $\theta_3 = d\varsigma - g\varsigma\overline{\theta}_1$ and $\Omega = \theta_1\theta_2\theta_3$. Thus Ω is a section of the canonical bundle of $M \times \mathbf{C}$. If $M \times \mathbf{C}$ can be realized by a hypersurface in $\mathbf{C}^3$ then, using (w_1, w_2, w_3) as coordinates on $\mathbf{C}^3$, $dw_1 dw_2 dw_3$ is also a section of this bundle and so is a multiple of Ω. Thus $d(f\Omega) = 0$ for some nonzero function f. But

$$d(f\Omega) = (Lf + fg + f_\varsigma g\varsigma)\overline{\theta}_1\Omega + f_{\overline{\varsigma}}\overline{\theta}_3\Omega.$$

In particular, we may set $\varsigma = 0$ to obtain $L(-\ln f) = g$. Thus if $M \times \mathbf{C}$ is realizable then $Lf = g$ is solvable. Conversely, if $Lf = g$ then $(z_1, z_2, \varsigma e^{-f})$ provides an embedding into $\mathbf{C}^3$.

It is simple to give a generalization to higher dimensions. Let M^{2n+1} be a hypersurface in $\mathbf{C}^{n+1}$. Let g be a 1-form on M which is a representative of a $\overline{\partial}_b$-cohomology class $[g]$ in $H^{0,1}(M)$. Let $(z_1, \ldots, z_{n+1})$ be coordinates on $\mathbf{C}^{n+1}$

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and ς a coordinate on $\mathbf{C}$. Note that $dz_1 \cdots dz_{n+1}$ is nonzero when restricted to M. Define a CR structure on $M \times \mathbf{C}$ by setting

$$\theta_j = dz_j, \qquad j = 1, \ldots, n+1, \quad \text{and} \quad \theta_{n+2} = d\varsigma - \varsigma g.$$

THEOREM. *This CR structure is integrable and depends only on* $[g]$. *It is nonrealizable precisely when* $[g] \neq 0$.

REMARK. If M is nondegenerate hypersurface of "signature" $(1, n-1)$ then $H^{0,1}$ is not zero and so there exists a degenerate nonrealizable CR structure of signature $(1, n-1, 0)$.

PROOF. Since g is taken to represent a cohomology class we have that $dg = \sum_1^{n+1} g_j \wedge \theta_j$. Thus $d\theta_{n+2} = -d\varsigma \wedge g - \varsigma dg = -\theta_{n+2} \wedge g - \varsigma(g_j \wedge \theta_j)$. So $d\theta_j \in \{\theta_1, \ldots, \theta_{n+2}\}$ and our structure is integrable. Let g and h represent the same class in $H^{0,1}$. So $g = h + \sum_1^{n+1} \alpha_j \theta_j + df$ for some functions on M. Let $\omega = d\eta - \eta h$ for η a complex variable. We claim that $S = \{\theta_1, \ldots \theta_{n+2}\}$ and $S' = \{\theta_1 \cdots \theta_{n+1}, \omega\}$ define the same CR structure. By this we mean there is a map $\Phi\colon M \times \mathbf{C} \to M \times \mathbf{C}$ such that $\Phi_*(\theta_j) \in$ linear span S'. Let $\Phi(z, \eta) = (z, \eta e^f)$. So $\Phi_* \theta_j = \theta_j$, $j = 1, \ldots, n+1$, and

$$\Phi_*(\theta_{n+2}) = d(\eta e^f) - \eta e^f g = e^f \omega \mod \{\theta_1, \ldots, \theta_{n+1}\}.$$

Thus $\Phi_*(\theta_{n+2}) \in S'$. In particular, if $[g] = 0$ then g and 0 define the same CR structure. This structure is now the product of the CR structures $M \times \mathbf{C}$ and hence is realizable in $\mathbf{C}^{n+2}$. Thus it only remains to show that if the CR structure given by S is realizable then $[g] = 0$. Let $\Omega = \theta_1 \cdots \theta_{n+2}$ be a section of the canonical bundle. Since $\theta_1 \cdots \theta_{n+1} dg = 0$, we have that $d\Omega = g \wedge \Omega$. But, by the same reasoning as before, if $M \times \mathbf{C}$ is realizable, then $d(f\Omega) = 0$ for some function f. Thus

$$(d'f + fg + f_\varsigma \varsigma g + f_{\bar\varsigma} \overline{\varsigma g} + f_{\bar\varsigma} \bar\theta_{n+2})\Omega = 0$$

where d' is with respect to M. Set $\varsigma = 0$ to obtain $(d'(-\ln f) - g)\theta_1 \cdots \theta_{n+1} = 0$. This implies $[g] = 0$.

The author would like to thank Claude LeBrun for bringing to his attention the particular formulation in [**E**] of our example (1). Our proof and generalization follow naturally from the knowledge that (1) is nonrealizable. The harder task, accomplished by Hill, Penrose, and Sparling, was to find this example.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, CAMDEN, NEW JERSEY 08102