SIMPLE EXAMPLES OF NONREALIZABLE CR HYPERSURFACES
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ABSTRACT. A new proof is provided of a nonrealizability result due to Hill, Penrose, and Sparling. This result is then generalized to higher dimensions: Each $\bar{\partial}_b$-cohomology class in $H^{0,1}(M)$ can be used to define a nonrealizable CR structure on $M \times \mathbb{R}^2$.

Consider an example due to Hill, Penrose, and Sparling [P] as formulated by Eastwood [E]. Let $M$ be a three-dimensional hypersurface in $\mathbb{C}^2$ with Lewy operator $L$. Define a CR structure on $M \times \mathbb{C}$ by taking

$$
L_1 = L + g\zeta \frac{\partial}{\partial \zeta}, \quad L_2 = \frac{\partial}{\partial \bar{\zeta}},
$$

where $g$ is a function on $M$ and $\zeta$ is the coordinate for $\mathbb{C}$. It is known that when $Lf = g$ has no solution, then this structure cannot be realized as a hypersurface in $\mathbb{C}^3$. The proof outlined in [P] uses the extension of holomorphic vector bundles across boundaries in the base. A simple proof using a Taylor series expansion was given in [E]. Here we give another simple proof; this one uses the canonical bundle $K$ of a CR manifold. This bundle has also been useful in other contexts [F, J]. If $M^{2n+1}$ is a CR manifold of hypersurface type and if $\theta_1, \theta_2, \ldots, \theta_{n+1}$ are independent forms each of which annihilates every Lewy vector field, then $K = \{ \lambda \theta_1 \wedge \cdots \wedge \theta_{n+1}, \lambda \in \mathbb{C}^* \}$. Note that $K$ does not depend on the particular choices of $\theta_j$.

So now let $\theta_1 = dz_1$ and $\theta_2 = dz_2$ where $(z_1, z_2)$ are the coordinates on $\mathbb{C}^2$ restricted to $M$. We may assume $dz_1 \bar{d}z_1 dz_2 \neq 0$ and also that $Lz_1 = 0, Lz_2 = 0, L\bar{z}_1 = 1$. Let $\theta_3 = d\zeta - g\zeta \partial_1$ and $\Omega = \theta_1 \theta_2 \theta_3$. Thus $\Omega$ is a section of the canonical bundle of $M \times \mathbb{C}$. If $M \times \mathbb{C}$ can be realized by a hypersurface in $\mathbb{C}^3$ then, using $(w_1, w_2, w_3)$ as coordinates on $\mathbb{C}^3$, $dw_1 dw_2 dw_3$ is also a section of this bundle and so is a multiple of $\Omega$. Thus $d(f \Omega) = 0$ for some nonzero function $f$. But

$$
d(f \Omega) = (Lf + fg + f_\zeta g_\zeta) \partial_1 \Omega + f_2 \bar{\theta}_3 \Omega.
$$

In particular, we may set $\zeta = 0$ to obtain $L(-\ln f) = g$. Thus if $M \times \mathbb{C}$ is realizable then $Lf = g$ is solvable. Conversely, if $Lf = g$ then $(z_1, z_2, \zeta e^{-f})$ provides an embedding into $\mathbb{C}^3$.

It is simple to give a generalization to higher dimensions. Let $M^{2n+1}$ be a hypersurface in $\mathbb{C}^{n+1}$. Let $g$ be a 1-form on $M$ which is a representative of a $\bar{\partial}_b$-cohomology class $[\gamma]$ in $H^{0,1}(M)$. Let $(z_1, \ldots, z_{n+1})$ be coordinates on $\mathbb{C}^{n+1}$.
and $\zeta$ a coordinate on $C$. Note that $dz_1 \cdots dz_{n+1}$ is nonzero when restricted to $M$. Define a CR structure on $M \times C$ by setting

$$\theta_j = dz_j, \quad j = 1, \ldots, n + 1, \quad \text{and} \quad \theta_{n+2} = d\zeta - \zeta g.$$  

**Theorem.** This CR structure is integrable and depends only on $[g]$. It is non-realizable precisely when $[g] \neq 0$.

**Remark.** If $M$ is nondegenerate hypersurface of “signature” $(1, n - 1)$ then $H^{0,1}$ is not zero and so there exists a degenerate nonrealizable CR structure of signature $(1, n - 1, 0)$.

**Proof.** Since $g$ is taken to represent a cohomology class we have that $dg = \sum_{i=1}^{n+1} g_j \wedge \theta_j$. Thus $d\theta_{n+2} = -d\zeta \wedge g - \zeta dg = -\theta_{n+2} \wedge g - \zeta (g_j \wedge \theta_j)$. So $d\theta_j \in \{\theta_1, \ldots, \theta_{n+2}\}$ and our structure is integrable. Let $g$ and $h$ represent the same class in $H^{0,1}$. So $g = h + \sum_{i=1}^{n+1} \alpha_i \theta_j + df$ for some functions on $M$. Let $\omega = d\eta - \eta h$ for $\eta$ a complex variable. We claim that $S = \{\theta_1, \ldots, \theta_{n+2}\}$ and $S' = \{\theta_1 \cdots \theta_{n+1}, \omega\}$ define the same CR structure. By this we mean there is a map $\Phi : M \times C \to M \times C$ such that $\Phi^* (\theta_j) \in \text{linear span } S'$. Let $\Phi (z, \eta) = (z, \eta e^f)$. So $\Phi^* \theta_j = \theta_j, \ j = 1, \ldots, n + 1$, and

$$\Phi^* (\theta_{n+2}) = d(\eta e^f) - \eta e^f d\eta = e^f \omega \mod \{\theta_1, \ldots, \theta_{n+1}\}.$$  

Thus $\Phi^* (\theta_{n+2}) \in S'$. In particular, if $[g] = 0$ then $g$ and $0$ define the same CR structure. This structure is now the product of the CR structures $M \times C$ and hence is realizable in $C^{n+2}$. Thus it only remains to show that if the CR structure given by $S$ is realizable then $[g] = 0$. Let $\Omega = \theta_1 \cdots \theta_{n+2}$ be a section of the canonical bundle. Since $\theta_1 \cdots \theta_{n+1} dg = 0$, we have that $d\Omega = g \wedge \Omega$. But, by the same reasoning as before, if $M \times C$ is realizable, then $d(f\Omega) = 0$ for some function $f$.

Thus

$$(d'f + fg + f \zeta g + f \zeta \bar{g} + f \bar{\theta}_{n+2}) \Omega = 0$$

where $d'$ is with respect to $M$. Set $\zeta = 0$ to obtain $(d'(-\ln f) - g)\theta_1 \cdots \theta_{n+1} = 0$. This implies $[g] = 0$.

The author would like to thank Claude LeBrun for bringing to his attention the particular formulation in [E] of our example (1). Our proof and generalization follow naturally from the knowledge that (1) is nonrealizable. The harder task, accomplished by Hill, Penrose, and Sparling, was to find this example.

**References**


