POLYNOMIAL RINGS OVER GOLDIE RINGS
ARE OFTEN GOLDIE
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ABSTRACT. Here, we prove a result that has as a consequence the fact that if the ring $R$ is an algebra over an uncountable field the a.c.c. on annihilators is preserved under polynomial extensions by any number of variables.

Recently Jeanne Kerr [1] has given an example of a ring $R$ with ascending chain condition on annihilators that has the property that the polynomial ring $R[X]$ does not have the ascending chain condition on annihilators (a.c.c.). This answered a question of long standing duration on the behavior of the classical Goldie conditions under polynomial extensions. Kerr’s example is an algebra over $\mathbb{Z}_2$.

**Lemma.** Let $R$ be a ring which contains an uncountable set $V$ in the center of $R$ having the property that if $u, v \in V$ and $u \neq v$, then $u - v$ is a nonzero divisor. Let $S$ be a countable subring of $R$. Then, there is an infinite subset $V'$ of $V$ such that $V'$ is algebraically independent over $S$.

**Proof.** Let $V'$ be a maximal subset of $V$ which is algebraically independent over $S$. Suppose $V'$ is finite. Then, any $v \in V$ satisfies a nonzero polynomial in the ring $(S[V'])[X]$. Since there are uncountably many elements in $V$ and only countably many polynomials, there is a polynomial $f(x)$ with infinitely many roots in $V$. Let $u$ and $v$ be two of these. Since we may divide by monics in any ring, $f(x) = (x - u)h(x)$, but $f(v) = 0 = (v - u)h(v)$ and $v - u$ is regular so $h(v) = 0$, and continuing, we obtain a contradiction since the number of roots is larger than the degree of $f(x)$.

**Theorem.** Let $R$ be a ring containing an uncountable set $V$ in the center of $R$, so that if $u$ and $v$ are distinct elements of $V$, $u - v$ is a nonzero divisor. Let $P$ be a property so that

\[ \text{(**)} \quad \text{A ring } T \text{ satisfies } P \text{ if and only if every countable subring of } T \text{ satisfies } P \text{ (e.g., the a.c.c. on annihilators).} \]

Then $R$ satisfies $P$ if and only if $R[X]$ satisfies $P$, where $X$ is any set of variables.

**Proof.** One way is trivial. Conversely, we show $R[X]$ satisfies $P$ by showing that every countable subring $R_0$ of $R[X]$ does. Clearly, $R_0 \subset S[x_1, x_2, \ldots]$ for some countable subring $S$ of $R$ and countable set of variables. Choose $V'$ as in the lemma so that $S[x_1, x_2, \ldots] \approx S[V'] \subset R$. Then, since $R_0 \subset S[x_1, x_2, \ldots]$, $R_0$ has $P$, so that by ** $R[X]$ does.
COROLLARY. If $R$ is an algebra over an uncountable field and $X$ is any set of variables then $R$ has the ascending chain condition on annihilators if and only if $R[X]$ does.

REMARK. The natural proof of the above, say in one variable, consists in the following: Any statement about chains of annihilators involves only countably many polynomials. Now, find an element in the field on which none of these vanish and embed the problem in $R$ by evaluation at this element.

Schock [2] proved that finite dimensionality is preserved by polynomial extensions, so that

COROLLARY. If $R$ is an algebra over an uncountable field, $R$ is Goldie if and only if $R[X]$ is Goldie.

REFERENCES


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