A NOTE ON THE AUTOMORPHISM GROUPS OF LIE MODULE TRIPLE SYSTEMS

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ABSTRACT. Using representation theory and well-known facts about automorphism groups of reductive Lie algebras, the automorphism group of a basic Lie module triple system \((M, \{ , , \}, L, b, \phi)\) over an algebraically closed field of characteristic zero is related to the automorphism group of the Lie algebra \(L\), the automorphism group of the standard embedding \(S\) of \((M, \{ , , \})\), and the automorphism group of the split null extension \(\tilde{S}\) of \(L\) by \(M\).

1. Introduction. The problem considered in this paper is that of determining the automorphism group \(\text{Aut}(M, \{ , , \})\) of a basic Lie module triple system (LMTS) \((M, \{ , , \}, L, b, \phi)\) over an algebraically closed field \(k\) of characteristic zero. We will do this by relating \(\text{Aut}(M, \{ , , \})\) to the automorphism group \(\text{Aut}L\) of the reductive Lie algebra \(L\) and using well-known facts about automorphism groups of reductive Lie algebras (see §2 for definitions).

The relationship between \(\text{Aut}(M, \{ , , \})\) and \(\text{Aut}L\) is seen by defining two algebra structures \(S\), the standard embedding of \((M, \{ , , \})\), and \(\tilde{S}\), the split null extension of \(L\) by \(M\). The groups \(\text{Aut}(S, \sigma)\) and \(\text{Aut}(\tilde{S}, \sigma)\) of automorphisms of \(S\) and \(\tilde{S}\), respectively, stabilizing \(M\) and \(L\) have natural restriction maps \(r : \text{Aut}(S, \sigma) \rightarrow \text{Aut}L\) and \(\tilde{r} : \text{Aut}(\tilde{S}, \sigma) \rightarrow \text{Aut}L\) which are homomorphisms. Since \(S\) is generated by \((M, \{ , , \})\) it is not surprising that \(\text{Aut}(M, \{ , , \}) \cong \text{Aut}(S, \sigma)\) and so \(\text{Aut}(M, \{ , , \})\) will be determined once the kernel and image of \(r\) are known. However, since \(\tilde{S}\) only involves the \(L\)-module structure of \(M\), the kernel and image of \(\tilde{r}\) are much easier to compute using standard representation theory facts (Theorems 3.1 and 3.2 and Lemma 4.1). Since \(\text{Aut}(S, \sigma) \subseteq \text{Aut}(\tilde{S}, \sigma)\), using Lemma 3.5, Corollary 3.6, and Lemma 4.2 it is possible to determine the kernel and image of \(r\) from those of \(\tilde{r}\) (Theorems 3.7 and 3.9 and Lemma 4.3). The surprising conclusion is that if \(\phi\) is symmetric or symplectic, the images of \(r\) and \(\tilde{r}\) are the same (Corollary 3.8), \(\text{Aut}(S, \sigma)\) is a normal subgroup of \(\text{Aut}(\tilde{S}, \sigma)\), and \(\text{Aut}(\tilde{S}, \sigma)/\text{Aut}(S, \sigma) \cong k^*\) (Theorem 4.4).

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2. Basic definitions. Recall from [3] that a Lie module triple system (abbreviated LMTS) is formed from a finite-dimensional Lie algebra \(L\) having a nondegenerate symmetric associative bilinear form \(b\) and a finite-dimensional faithful \(L\)-module \(M\) having a nondegenerate \(L\)-invariant bilinear form \(\phi\), that is

\[
\phi(xl, y) = -\phi(x, yl)
\]

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for all \(x, y \in M, l \in \mathcal{L}\). A triple product \(\{, , \}\) on \(M\) is defined by setting \(\{xyz\} := xR(y, z)\) where \(R: M \times M \to \mathcal{L}\) is given by

\[
(2.2) \quad b(l, R(y, z)) = \phi(zl, y)
\]

for all \(y, z \in M, l \in \mathcal{L}\). The *Lie module triple system* \((M, \{, , \})\) is denoted \((M, \{, , \}, \mathcal{L}, b, \phi)\) when the ingredients need to be specified.

We will always assume that \((M, \{, , \}, \mathcal{L}, b, \phi)\) is a simple LMTS such that \(M\) is completely reducible as an \(\mathcal{L}\)-module. Hence \(\mathcal{L}\) is reductive (Theorem 10, p. 81 of [4]), i.e. \(\mathcal{L} = [\mathcal{L}, \mathcal{L}] \oplus C\) with \([\mathcal{L}, \mathcal{L}]\) semisimple and \(C\) a central ideal of \(\mathcal{L}\) whose elements act semisimply on \(M\). \(\Lambda\) will denote the set of lowest weights, without multiplicities, of \(M\) as an \(\mathcal{L}\)-module. We will usually assume that \((M, \{, , \}, \mathcal{L}, b, \phi)\) is one of the following *basic* types of LMTS’s:

**Type (I).** \(M\) is an irreducible self-dual \(\mathcal{L}\)-module of lowest weight \(\lambda\), \(\mathcal{L}\) is semisimple, and \(\phi\) is symmetric or symplectic.

**Type (II).** \(M = M_1 \oplus M_2\) with \(M_1\) and \(M_2\) isomorphic irreducible self-dual \(\mathcal{L}\)-modules of lowest weight \(\lambda_1, \lambda_2\) respectively with \(\lambda_1 \neq \lambda_2\). \(\phi\) restricted to \(M_i \times M_i\) is identically zero for \(i = 1, 2\). \(\text{Der}(M, \{, , \}) = \mathcal{L} \oplus \text{sl}(2)\) where \(\text{Der}(M, \{, , \})\) is the derivation algebra of \((M, \{, , \})\) and \(\mathcal{L}, \text{sl}(2)\) are ideals of \(\text{Der}(M, \{\mathcal{L}, b, \phi\}).\)

**Type (III).** \(M = M_1 \oplus M_2\) with \(M_1\) and \(M_2\) irreducible dual \(\mathcal{L}\)-submodules of lowest weight \(\lambda_1\) and \(\lambda_2\) respectively with \(\lambda_1 \neq \lambda_2\). \(\phi\) restricted to \(M_i \times M_i\) is identically zero and there is a \(z \in k^\ast\) with \(\phi(z, y) = a\phi(y, z)\) for all \(z \in M_1, y \in M_2\). \(\mathcal{L}\) is semisimple or has a one-dimensional center spanned by \(c\) where \((x_1 + x_2)c := x_1 - x_2\) for \(x_i \in M_i\).

Note that simple Lie triple systems are basic [5, 1]. Other examples of basic LMTS’s are given in [1, 2] where it was shown that if \((M, \{, , \}, \mathcal{L}, b, \phi)\) is a LMTS for which \(M\) is a completely reducible \(\mathcal{L}\)-module and \(\phi\) is symmetric or symplectic, then \((M, \{, , \})\) can be constructed from basic LMTS’s and one- and two-dimensional abelian LMTS’s. Note however that not all simple completely reducible LMTS’s are basic (see [1] for examples of nonbasic simple LMTS’s).

\(\alpha \in \text{GL}(M)\) is an automorphism of \((M, \{, , \})\) if \(\{xyz\} \alpha = \{x\alpha, y\alpha, z\alpha\}\) and \(\text{Aut}(M, \{, , \})\) will denote the automorphism group of \((M, \{, , \}, \mathcal{L}, b, \phi)\). There are two nonassociative algebra structures naturally defined on \(M \oplus \mathcal{L}\), the first being the split null extension \(\mathcal{S}\) of \(\mathcal{L}\) by \(M\), with multiplication defined by \((x + l)(y + l_1) := xl_1 - yl + [l, l_1]\), and the second being the *standard embedding* \(\mathcal{S}\) of \((M, \{, , \})\) (see [3]), with multiplication defined by \((x + l)(y + l_1) := xl_1 - yl + [l, l_1] + R(x, y)\) for \(x, y \in M, l, l_1 \in \mathcal{S}\). \(\sigma \in \text{GL}(M \oplus \mathcal{L})\) defined by \((x + l)\sigma := -x + l\) is an automorphism of both \(\mathcal{S}\) and \(\mathcal{S}\) fixing \(\mathcal{L}\). Let \(\text{Aut}(\mathcal{S}, \sigma)\) (respectively \(\text{Aut}(\mathcal{S}, \sigma)\)) be the automorphisms of \(\mathcal{S}\) (respectively \(\mathcal{S}\)) which commute with \(\sigma\).

Clearly \(\text{Aut}(\mathcal{S}, \sigma)\) is a subgroup of \(\text{Aut}(\mathcal{S}, \sigma)\) and \(\psi: \text{Aut}(M, \{, , \}) \to \text{Aut}(\mathcal{S}, \sigma)\) defined by \((x + l)(\tau\psi) := x\tau + (\tau^{-1}l\tau)\) for \(\tau \in \text{Aut}(M, \{, , \})\) is an isomorphism. For computing \(\text{Aut}(\mathcal{S}, \sigma)/\text{Aut}(\mathcal{S}, \sigma)\) we will also need to consider the automorphism group \(\text{Aut}(\mathcal{L}, \mathcal{L})\) of \(\mathcal{L}\) and the homomorphisms \(r: \text{Aut}(\mathcal{S}, \sigma) \to \text{Aut}(\mathcal{L}, \mathcal{L})\) and \(\bar{r}: \text{Aut}(\mathcal{S}, \sigma) \to \text{Aut}(\mathcal{L}, \mathcal{L})\) given by restriction, i.e. \(\tau_r := \tau|_\mathcal{L}\) and \(\eta_{\bar{r}} := \eta|_\mathcal{L}\) for \(\tau \in \text{Aut}(\mathcal{S}, \sigma), \eta \in \text{Aut}(\mathcal{S}, \sigma)\). The kernel of \(r\) (respectively \(\bar{r}\)) is denoted \(\ker r\) (respectively \(\ker \bar{r}\)) and the image of \(r\) (respectively \(\bar{r}\)) by \(\text{im } r\) (respectively \(\text{im } \bar{r}\)).
3. Images. Since any automorphism of \( L \) is the product of an inner automorphism and one that stabilizes a Borel subalgebra of \( L \) [4], we will consider these two types of automorphisms of \( L \) separately. \( \text{Inn Aut} L \) will be the group of inner automorphisms of \( L \). For \( e \in L \) nilpotent define \( \exp(\text{ad}se) \in \text{Aut}(S, \sigma) \) by
\[
\exp(\text{ad}se) := \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}se)^n \quad \text{where} \quad x(\text{ad}se) := xe \quad \text{for all} \quad x \in S.
\]

**Theorem 3.1** (Corollary 5, p. 44 of [6]). Let \( \text{Inn Aut}(S, \sigma) \) be the subgroup of \( \text{Aut}(S, \sigma) \) generated by \( \{\exp(\text{ad}se) | e \in L, e \text{ nilpotent}\} \). Then \( \tau r \in \text{Inn Aut} L \) for \( \tau \in \text{Inn Aut}(S, \sigma) \), \( r : \text{Inn Aut}(S, \sigma) \to \text{Inn Aut} L \) is onto, and its kernel is isomorphic to \( \Lambda_w/\Lambda_r \) where \( \Lambda_r \) is the root lattice of \( [L, L] \) and \( \Lambda_w \) is the weight lattice of \( M \) as an \( [L, L] \)-module. Thus \( \text{Inn Aut} L \subseteq \text{im} \tau \subseteq \text{im} r \).

Thus if \( \nu \in \text{Aut}(S, \sigma) \), \( \nu = \eta_1 \eta_2 \) where \( \eta_1 \in \text{Inn Aut}(S, \sigma) \) and \( \eta_2 \tau \) stabilizes a Borel subalgebra of \( L \).

Now suppose \( \tau \in \text{Aut} L \) stabilizes a Borel subalgebra \( B \) of \( L \). Then it stabilizes the Cartan subalgebra \( H \) contained in \( B \), and if \( \tau^*: H^* \to H^* \) is defined by \( h(\mu \tau^*) := \mu(h^{-1} r^{-1}) \) for \( h \in H \), \( \mu \in H^* \), then \( \alpha \tau^* \) is a positive root if \( \alpha \) is a positive root and \( \Delta \tau^* = \Delta \) for \( \Delta \) a base of the positive roots where \( \Delta \tau^* := \{\alpha \tau^* | \alpha \in \Delta\} \).

**Theorem 3.2.** Suppose \( \tau \in \text{Aut} L \) stabilizes a Borel subalgebra \( B \) of \( L \) and \( N_i \) is the \( \lambda_i \) weight space relative to \( B \) for \( \lambda_i \in \Lambda = \{\lambda_1, \ldots, \lambda_l\} \). Then \( \tau \in \text{im} \tau \) if and only if \( \lambda_i \tau^* = \lambda_i \) and \( \dim N_i = \dim N_j \) if \( \lambda_i \tau^* = \lambda_j \).

**Proof.** Suppose \( \eta \in \text{Aut}(S, \sigma) \) with \( \eta \tau = \tau \). Then if \( \mu \) is a weight of \( M \) as an \( L \)-module and \( M_\mu \) is the \( \mu \) weight space of \( M \), \( x \in M_\mu \). Since \( \tau \) permutes positive root spaces, \( x \eta \) is a lowest weight vector if and only if \( x \) is a lowest weight vector. Thus \( \Lambda \tau^* = \Lambda \) and \( \dim N_i = \dim N_j \) if \( \lambda_i \tau^* = \lambda_j \).

Conversely, suppose \( \tau \in \text{Aut} L \) stabilizes \( B \), \( \Lambda \tau^* = \Lambda \) and \( \dim N_i = \dim N_j \) if \( \lambda_i \tau^* = \lambda_j \). Define a new action of \( L \) on \( M \) by \( x : l := x(l \tau) \) and denote \( M \) with this new \( L \)-action by \( \tilde{M} \). Now if \( x \in M_\mu \), then \( x \in \tilde{M}_{\mu \tau} \) so \( M \) and \( \tilde{M} \) have the same lowest weights. Since the \( \lambda_i \) weight spaces of \( M \) and \( \tilde{M} \) are the same dimension for \( \lambda_i \in \Lambda \), \( M \) and \( \tilde{M} \) are isomorphic \( L \)-modules, i.e. there is a linear bijection \( \nu : M \to \tilde{M} \) with \( (x + l) \nu = (x \nu) \cdot l = (x \nu)(l \tau) \). Thus \( \eta \in \text{GL}(M + L) \) defined by \( (x + l) \eta := x \nu + l \tau \) is in \( \text{Aut}(S, \sigma) \) and \( \eta \tau = \tau \). Hence \( \tau \in \text{im} \tau \).

**Note 3.3.** If \( \eta \in \text{Aut}(S, \sigma) \) with \( \eta \tau = \tau \) stabilizing a Borel subalgebra \( B \) of \( L \), then \( y \eta \) is a highest weight vector of \( M \) if and only if \( y \) is a highest weight vector. In particular, if \( x_1 \) and \( x_2 \) are nonzero lowest weight vectors with \( x_1 \eta = x_2 \), let \( y_1 = x_1 \cdot e_1 \cdots e_n \) be a nonzero highest weight vector and let \( y_2 \) be any nonzero highest weight vector of the irreducible submodule generated by \( x_2 \). Then \( y_1 \eta = (x_1 \eta)(e_1 \tau) \cdots (e_n \tau) = cy_2 \) for some \( c \in k \) and \( c \neq 0 \) since \( \eta \) is injective.

It is also important to note that if \( \{x_{i,1}, \ldots, x_{i,k}\} \) is a basis of \( N_i \) and \( \{x'_{j,1}, \ldots, x'_{j,k}\} \) is a basis of \( N_j \), then the \( \eta \) constructed in the second half of the proof above can be defined so that \( x_{i,n} \eta = x'_{j,n} \) for \( n = 1, \ldots, k \) and \( i = 1, \ldots, l \). For if \( M_i,n \) is the irreducible submodule of \( M \) generated by \( x_{i,n} \) and \( \tilde{M}_{j,n} \) is the irreducible submodule of \( \tilde{M} \) generated by \( x'_{j,n} \), then \( M_i,n \) and \( \tilde{M}_{j,n} \) are isomorphic \( L \)-modules since \( \lambda_i \tau^* = \lambda_j \). Let \( \nu_{i,n} : M_i,n \to \tilde{M}_{j,n} \) be such an isomorphism. Adjusting
by a scalar multiple if necessary, we may assume $x_i,n \nu_i,n = x'_i,n$. Letting $M_i := \sum_{n=1}^{k} M_i,n$ and $\tilde{M}_i := \sum_{n=1}^{k} \tilde{M}_i,n$, define $\nu_i : M_i \rightarrow \tilde{M}_i$ by $(\sum_{n=1}^{k} z_n)\nu_i := \sum_{n=1}^{k} (z_n \nu_i, n)$ for $z_n \in M_i,n$ and define $\nu : M \rightarrow \tilde{M}$ by $(\sum_{i=1}^{k} z_i)\nu := \sum_{i=1}^{k} (z_i, \nu_i)$. As before $\eta$ is defined by $(x + l)\eta := z\nu + lr$.

Note 3.4. Finally note that if $\eta \in \text{Aut}(\mathfrak{S}, \sigma)$, then $\eta$ is completely determined by $\eta \tilde{r} = r \in \text{Aut} \tilde{L}$ and the action of $\eta$ on a set of vectors $\{x_1, \ldots, x_j\}$ spanning the lowest weight spaces of $M$ since vectors of the form $x_1,l_1 \cdots l_n$ span $M$ and $(x_1,l_1 \cdots l_n) = (x_{\tilde{r}})(lr) \cdots (l_n r)$.

Having determined $\text{im} \tilde{r}$, the following lemma will allow us to determine $\text{im} r$ since $\text{im} \tilde{r} \subseteq \text{im} r$. Note that the lemma is true for any LMTS.

**Lemma 3.5.** Suppose $\eta \in \text{Aut}(\mathfrak{S}, \sigma)$. Then $\eta \in \text{Aut}(\mathfrak{S}, \sigma)$ if and only if for all $l \in \mathcal{L}$, $y, z \in M$

\[ b(l\eta, R(y, z)\eta) = \phi((zl)\eta, y\eta). \]

**Proof.** Suppose $\eta \in \text{Aut}(\mathfrak{S}, \sigma)$. Then $R(y, z)\eta = R(y\eta, z\eta)$ so by (2.2)

\[ b(l\eta, R(y, z)\eta) = b(l\eta, R(y\eta, z\eta)) = \phi((zl)\eta, y\eta) = \phi((zl)\eta, y\eta) \]

giving (3.5.1). Conversely, suppose $\eta \in \text{Aut}(\mathfrak{S}, \sigma)$ satisfies (3.5.1). Then for $u, v \in M$

\[ b(R(u, v)\eta, R(y, z)\eta) = \phi([zR(u, v)\eta], y\eta) = \phi((zl)\eta, R(y, z)\eta) \]

by (2.2) so $R(y, z)\eta = R(y\eta, z\eta)$ and $\eta \in \text{Aut}(\mathfrak{S}, \sigma)$.

**Corollary 3.6.** Suppose $(M, \{ , , \})$ is basic and $\eta \in \text{Aut}(\mathfrak{S}, \sigma)$. Then $\eta \in \text{Aut}(\mathfrak{S}, \sigma)$ if and only if

\[ \phi(z\eta, y\eta) = \phi(z, y) \]

for all $y, z \in M$.

**Proof.** If $\mathcal{L}$ is semisimple, then $b(lr, l_1 r) = b(l, l_1)$ for all $r \in \text{Aut} \mathcal{L}$, $l, l_1 \in \mathcal{L}$ [4]. Thus $b(l\eta, l_1 \eta) = b(l, l_1)$ for all $l, l_1 \in \mathcal{L}$ if $(M, \{ , , \})$ is Type (I) or (II) or if $(M, \{ , , \})$ is Type (III) with $\mathcal{L}$ semisimple. If $(M, \{ , , \})$ is Type (III) with $\mathcal{L}$ having a one-dimensional center, Theorem 3.2 implies that either $M_1 \eta = M_2$ and $M_2 \eta = M_1$ or $M_1 \eta = M_1$ and $M_2 \eta = M_2$. In the first case $c\eta = -c$ and in the second case $c\eta = c$ so $b(c\eta, c\eta) = b(c, c)$ and $b(l\eta, l_1 \eta) = b(l, l_1)$ for all $l \in \mathcal{L}$ for $(M, \{ , , \})$ any basic LMTS. Then (3.5.1) becomes $\phi(zl\eta, y\eta) = b(l\eta, R(y, z)\eta) = b(l, R(y, z)) = \phi(zl, y)$ by (2.2) giving (3.6.1) since $M$ is spanned by vectors of the form $zl$.

Since $\text{Inn} \text{Aut} \mathcal{L} \subseteq \text{im} r$, we again only need to determine which $r \in \text{Aut} \mathcal{L}$ stabilizing a Borel subalgebra $B$ are in $\text{im} r$.

**Theorem 3.7.** Suppose $(M, \{ , , \})$ is a basic LMTS and $r \in \text{im} \tilde{r}$ stabilizes a Borel subalgebra $B$ of $\mathcal{L}$.

(i) If $\lambda r^* = \lambda$ for all $\lambda \in \Lambda$, then $r \in \text{im} r$.

(ii) If $(M, \{ , , \})$ is Type (III) with $a = \pm 1$ and $\lambda_1 r^* = \lambda_2$, then $r \in \text{im} r$.

**Proof.** We want to find $\eta \in \text{Aut}(\mathfrak{S}, \sigma)$ with $\eta \tilde{r} = r$ and such that (3.6.1) is satisfied. By Note 3.4 we can do this simply by specifying the images of the lowest weight vectors.
Suppose $\lambda \tau^* = \lambda$. If $(M, \{ , , \})$ is Type (I) let $x$ be a nonzero lowest weight vector and $y := xe_1 \cdots e_n$ be a nonzero highest weight vector. By Note 3.3, $x(e_1 \tau) \cdots (e_n \tau) = cy$ for some $c \in k^*$. By Notes 3.3 and 3.4 we can define $\eta \in \text{Aut}(\mathcal{S}, \sigma)$ by $l\eta := l\tau$ and $x\eta := \sqrt{c}^{-1}x$. Then $y\eta = \sqrt{c}y$ and $\phi(x\eta, y\eta) = \phi(x, y)$. If $w$ is a weight vector of $M$ of weight $\mu \neq -\lambda$, by (2.1) $\phi(x, w) = 0 = \phi(x\eta, w\eta)$ since $w\eta$ has weight $\mu \tau^* \neq -\lambda$. Thus $\phi(x\eta, w\eta) = \phi(x, w)$ for all $w \in M$. By induction on $m$

$$
\phi(xl_1 \cdots l_m\eta, w\eta) = \phi((x\eta)(l_1\eta) \cdots (l_m\eta), w\eta) \n
= -\phi((x\eta)(l_1\eta) \cdots (l_{m-1}\eta), (w\eta)(l_m\eta)) = -\phi(xl_1 \cdots l_{m-1}\eta, (wl_m)\eta, w) $$

so $\eta$ satisfies (3.6.1) and hence $\eta \in \text{Aut}(\mathcal{S}, \sigma)$ so $\tau \in \text{im}\tau$. If $(M, \{ , , \})$ is Type (II) or (III) Note 3.3 again shows that if $x_2$ is a nonzero lowest weight vector of $M_2$ and $y_2 = x_2e_1 \cdots e_n$ is a nonzero highest weight vector, then $x_2(e_1 \tau) \cdots (e_n \tau) = cy_2$ for some $c \in k^*$. Let $x_1$ be a nonzero lowest weight vector of $M_1$ and define $\eta \in \text{Aut}(\mathcal{S}, \sigma)$ by $x_1\eta := x_1$ and $x_2\eta := c^{-1}x_2$. Thus $y_2\eta = y_2$ so $\phi(x_1\eta, y_2\eta) = \phi(x_1, y_2)$ and $\phi(y_2\eta, x_1\eta) = \phi(y_2, x_1)$. By the same induction argument as before $\phi(z\eta, w\eta) = \phi(z, w)$ and $\phi(w\eta, z\eta) = \phi(w, z)$ for all $z \in M_1, w \in M_2$ giving (3.6.1).

For (ii) let $x_i \in M_i$ be nonzero lowest weight vectors for $i = 1, 2$ and $y_i \in M_i$ be nonzero highest weight vectors for $i = 1, 2$ with $y_2 = x_2e_1 \cdots e_n$. Then by Note 3.3, $x_1(e_1 \tau) \cdots (e_n \tau) = cy_1$ for some $c \in k^*$. Also $\phi(x_1, y_2) = d\phi(x_2, y_1)$ so if we define $\eta \in \text{Aut}(\mathcal{S}, \sigma)$ by $x_1\eta := x_1$ and $x_2\eta := c^{-1}x_1$, then $y_2\eta = dy_1$ and $\phi(x_1\eta, y_2\eta) = d\phi(x_2, y_1) = \phi(x_1, y_2)$ and again using induction we have $\phi(z\eta, w\eta) = \phi(z, w)$ for all $z \in M_1, w \in M_2$. Since $\phi$ is symmetric or symplectic we also have $\phi(w\eta, z\eta) = \phi(w, z)$ for $z \in M_1, w \in M_2$ so $\eta$ satisfies (3.6.1).

**Corollary 3.8.** Suppose $(M, \{ , , \}, \mathcal{L}, b, \phi)$ is a basic LMTS and $\phi$ is symmetric or symplectic. Then $\text{im}\tau = \text{im}\tau$. Hence $\text{Aut}(\mathcal{S}, \sigma) = \ker\bar{\tau}\text{Aut}(\mathcal{S}, \sigma)$.

**Proof.** The first statement is an immediate consequence of Theorems 3.1, 3.2 and 3.7. For the second one if $\eta \in \text{Aut}(\mathcal{S}, \sigma)$, there is an $\eta_1 \in \text{Aut}(\mathcal{S}, \sigma)$ with $\eta\bar{\tau} = \eta_1\tau = \eta_1\bar{\tau}$ so $\eta\bar{\tau}^{-1} \in \ker\bar{\tau}$

Thus the difference between $\text{Aut}(\mathcal{S}, \sigma)$ and $\text{Aut}(\mathcal{S}, \sigma)$ lies in $\ker\bar{\tau}$ if $(M, \{ , , \}, \mathcal{L}, b, \phi)$ is basic and $\phi$ is symmetric or symplectic. The following theorem finishes the question of determining $\text{im}\tau$ for $(M, \{ , , \})$ a basic LMTS.

**Theorem 3.9.** Suppose $(M, \{ , , \})$ is Type (III) with $a \neq \pm 1$ and $\tau \in \text{im}\tau$ stabilizes a Borel subalgebra of $\mathcal{L}$. Then $\tau \in \text{im}\tau$ if and only if $\lambda_1\tau^* = \lambda_i$ for $i = 1, 2$.

**Proof.** If $\lambda_i\tau^* = \lambda_i$ for $i = 1, 2 \tau \in \text{im}\tau$ by Theorem 3.7(i). If $\lambda_i\tau^* \neq \lambda_i$ for $i = 1, 2$, then $\lambda_1\tau^* = \lambda_2$ and $\lambda_2\tau^* = \lambda_1$ by Theorem 3.2, so if $\eta \in \text{Aut}(\mathcal{S}, \sigma)$ with $\eta\tau = \tau, M_1\eta = M_2$ and $M_2\eta = M_1$. But $\eta$ must satisfy (3.6.1), so for $y \in M_2, z \in M_1, \phi(y, z) = \phi(y\eta, z\eta) = a\phi(z\eta, y\eta) = a\phi(z, y) = a^2\phi(y, z)$. Thus $a^2 = 1$, i.e. $a = \pm 1$, a contradiction. Hence no such $\eta$ exists.
4. Kernels. In this section we will determine \( \ker \tau \) and \( \ker r \).

**Lemma 4.1.** Suppose \( \Lambda = \{\lambda_1, \ldots, \lambda_l\} \), \( N_i \) is the \( \lambda_i \) weight space of \( M \) for \( i = 1, \ldots, l \), and \( P = N_1 \oplus \cdots \oplus N_l \). Then \( N_i \) is \( \ker \tau \) invariant and if \( \dim N_i = n_i \) for \( i = 1, \ldots, l \), then \( \psi: \ker \tau \rightarrow GL(n_1, k) \times \cdots \times GL(n_l, k) \) defined by \( \eta \psi := \eta|_p \) is an isomorphism.

**Proof.** This follows from the proof of Theorem 3.2 and Notes 3.3 and 3.4 since if \( \eta \in \ker \tau \), \( \eta \) certainly fixes a Borel subalgebra of \( \mathcal{L} \).

**Lemma 4.2.** Suppose \( \eta \in \ker \tau \). Then \( \eta \in \ker r \) if and only if \( \eta \) satisfies (3.6.1).

**Proof.** If \( \eta \in \ker \tau \), \( \eta \eta_l = l \) for all \( l \in \mathcal{L} \), so \( b(l \eta_l, l_1 \eta) = b(l, l_1) \) for all \( l, l_1 \). The rest of the proof is the same as that of Corollary 3.6 using Lemma 4.1 of [3].

**Lemma 4.3.** (i) If \( (M\{\cdot,\cdot\}) \) is Type (I), \( \ker \tau \cong k^* \) and \( \ker r \cong C_2 \), the cyclic group of order 2.

(ii) If \( (M\{\cdot,\cdot\}) \) is Type (II), \( \ker \tau \cong GL(2, k) \) and \( \ker r \cong SL(2, k) \).

(iii) If \( (M\{\cdot,\cdot\}) \) is Type (III), \( \ker \tau \cong k^* \times k^* \) and \( \ker r \cong k^* \).

**Proof.** The assertions about \( \ker \tau \) follow from Lemma 4.1 and those about \( \ker r \) for Types (I) and (III) follow immediately from Lemma 4.2. For Type (II) recall that there is an ideal \( K \) of \( \text{Der}(M\{\cdot,\cdot\}) \) isomorphic to \( \text{sl}(2) \) and consisting of endomorphisms of \( M \) which commute with \( \mathcal{L} \), so \( PD \subseteq P \) for \( D \in K \) and \( P \) the lowest weight space of \( M \). Now if \( D \in K \), \( \phi(zD, y) = -\phi(x, yD) \) by Lemma 2.3 of [2]. Hence if \( D \in K \) is nilpotent, \( \exp D \in \text{Aut}(M\{\cdot,\cdot\}) \) where \( \exp D := \text{id} + D \), so \( \eta \eta_D \in \ker r \) where \( (x + l)\eta_D := \exp D + l \). Note that the group generated by \( \{\eta_D|D \in K \text{ nilpotent}\} \) is isomorphic to \( \text{SL}(2, k) \subseteq \text{GL}(2, k) = \ker \tau \) under restriction to \( P \). Now \( \text{GL}(2, k) = k^* \text{SL}(2) \) where \( \eta \in k^* \) for \( b \in k^* \) is defined by \( (x + l)\eta_b := bx + l \). Then Lemma 4.2 gives that \( \eta \in \ker r \) if and only if \( b = \pm 1 \).

**Theorem 4.4.** Suppose \( (M\{\cdot,\cdot\}, L, b, \phi) \) is a basic \( \text{LMTS} \) and \( \phi \) is symmetric or symplectic. Then \( \text{Aut}(S, \sigma) \) is a normal subgroup of \( \text{Aut}(\overline{S}, \sigma) \) and

\[
\text{Aut}(\overline{S}, \sigma)/\text{Aut}(S, \sigma) \cong k^*.
\]

**Proof.** For each type \( \ker \tau = k^* \ker r \) where for Types (I) and (III) \( \eta \eta_b \in k^* \) is defined for \( b \in k^* \) by \( (x + l)\eta_b := bx + l \) and for Type (II) \( \eta \eta_b \) is defined by \( (x_1 + x_2 + l)\eta_b := bx_1 + x_2 + l \) for \( x_i \in M_i \). Hence by Corollary 3.8, \( \text{Aut}(\overline{S}, \sigma) = \ker \overline{\text{Aut}}(S, \sigma) = k^* \text{Aut}(S, \sigma) \). For Types (I) and (II) \( \eta^{-1} \nu \eta_b = \nu \) for \( \nu \in \text{Aut}(S, \sigma) \) and for Type (III) if \( \nu \in \text{Aut}(S, \sigma) \) with \( M_i \nu = M_i, i = 1, 2 \), then \( \eta^{-1} \nu \eta_b = \nu \) and if \( \nu \in \text{Aut}(S, \sigma) \) with \( M_i \nu = M_2, M_2 \nu = M_1 \), then \( \eta^{-1} \nu \eta_b = \nu \xi \) where \( (x_1 + x_2 + l)\xi := bx_1 + b^{-1}x_2 + l \). Then \( \xi \in \text{Aut}(S, \sigma) \) so \( \text{Aut}(S, \sigma) \) is a normal subgroup in all three cases.

**Corollary 4.5.** Suppose \( (M\{\cdot,\cdot\}, L, b, \phi) \) is a basic \( \text{LMTS} \) with \( \phi \) symmetric or symplectic. If \( \Lambda_w \subseteq \Lambda_r \), \( \text{Aut}(S, \sigma) \cong \ker \tau \times \text{im} \tau \).

**Proof.** This follows from Theorem 3.1.

**Note 4.5.** When studying the forms of a Type (III) \( \text{LMTS} \) \( (M\{\cdot,\cdot\}, L, b, \phi) \) in a subsequent paper, we will need to consider the groups \( \text{Aut}^*(\overline{S}, \sigma) := \{\eta \in \text{Aut}(\overline{S}, \sigma)|M_i \eta = M_i \text{ for } i = 1, 2\} \) and \( \text{Aut}^*(S, \sigma) := \text{Aut}(S, \sigma) \cap \text{Aut}^*(\overline{S}, \sigma) \) and also the homomorphisms \( \overline{\tau}^*: \text{Aut}^*(\overline{S}, \sigma) \rightarrow \text{Aut}L \) and \( \tau^*: \text{Aut}^*(S, \sigma) \rightarrow \text{Aut}L \).
given by restriction. The results obtained already give the following statements about these groups: \( \ker^* \cong k^* \), \( \ker^\sigma \cong k^* \times k^* \), \( \text{Inn} \text{Aut}_L \subseteq \text{im}^* = \text{im}^\sigma \), and \( \text{Aut}^*(S, \sigma) \) is a normal subgroup of \( \text{Aut}^*(S, \sigma) / \text{Aut}^*(S, \sigma) \cong k^* \). If \( \eta \in \text{Aut}^*(S, \sigma) \) such that \( \eta \) stabilizes a Borel subalgebra \( B \), then \( \lambda_i \eta^* = \lambda_i \) for \( i = 1, 2 \). Either \( \text{Aut}^*(S, \sigma) = \text{Aut}(S, \sigma) \) or \( \text{Aut}^*(S, \sigma) \) is a normal subgroup of \( \text{Aut}(S, \sigma) \) of index 2. If \( a \neq \pm 1 \), \( \text{Aut}^*(S, \sigma) = \text{Aut}(S, \sigma) \).

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