ON TWO THEOREMS OF THOMPSON
ZHANG GUANGXIANG

ABSTRACT. Let $G$ be a finite group.

**THEOREM.** Let $P \in \text{Syl}_p(G)$ with $\Omega_1(P) \leq Z(P)$. If $N_G(Z(P))$ has a normal $p$-complement, then so does $G$.

**COROLLARY.** Let $M$ be a nilpotent maximal subgroup of $G$ and $P \in \text{Syl}_2(M)$ with $\Omega_2(P) \leq Z(P)$. Then $G$ is solvable.

This extends Thompson's solvability theorem [9]. We also give two other results generalizing Thompson's theorem.

In this note, we prove some theorems, one of which is similar to Thompson's normal $p$-complement theorem [8] and the others are generalizations of Thompson's theorem regarding solvability of finite groups [9].

For simplicity we write $X \in N_P$, which means the finite group $X$ has a normal $p$-complement for a prime $p$. Our other notations are standard and follow [5].

**THEOREM 1.** Let $G$ be a finite group and $P \in \text{Syl}_p(G)$. If $\Omega_1(P) \leq Z(P)$ and $N_G(Z(P)), C_G(Z(P)) \in N_P$, then $G \in N_P$.

**PROOF.** Let $G$ be a counterexample of minimal order of the theorem. Then we deduce a contradiction step-by-step.

1. $O_p'(G) = 1$ by the minimality of $|G|$.
2. $O_p(G) \neq 1$, since otherwise we would have $N_G(P_1) < G$ for each subgroup of $P_1$ of $P$ and $N_G(P_1) \in N_P$. In fact, we have a series of subgroups

$$N_G(P_1), N_G(P_2), \ldots, N_G(P_i), \ldots, N_G(P),$$

where $P_{i+1} \in \text{Syl}_p(N_G(P_i))$. Now suppose $N_G(P_i) \notin N_P$ but $N_G(P) \in N_P$. There is some $i_0$ such that $N_G(P_{i_0}) \notin N_P$ while $N_G(P_{i_0+1}) \in N_P$. Then we have

$$\Omega_1(P_{i_0+1}) \leq \Omega_1(P) \leq Z(P) \leq Z(P_{i_0+1}),$$

$$C_G(Z(P_{i_0+1})) \leq C_G(Z(P)) \in N_P.$$  

Thus $N_G(P_{i_0}) \in N_P$ by the minimality of $|G|$. This is a contradiction. So $N_G(P_1) \in N_P$ for each $P_1 \leq P$ and hence $G \in N_P$ by Frobenius' theorem [5, Theorem 7.45]. This contradicts our hypothesis of $G$.

3. Set $H = O_p(G)$. Then $G/H \in N_P$ by a discussion similar to step 2. Let $K/H$ be the normal $p$-complement of $G/H$. Then $1 < K < G$ and $G$ is $p$-solvable. Since $O_p'(G) = 1$, $C_G(H) \leq H$ by Theorem 6.3.2 of [5]. Particularly $Z(P) \leq H$.

4. Let $M$ be a maximal subgroup of $G$ containing $P$. Then $M \in N_P$ and $O_p'(M)H = O_p'(M) \times H$. This shows $O_p'(M) \leq C_G(H) \leq H$ and so $O_p'(M) = 1$.


This forces \( P = M \). Now we conclude \( C_G(Z(P)) = P \), for otherwise \( C_G(Z(P)) = G \in N_p \).

5. Since \( \Omega_1(P) \leq Z(P) \leq Z(H) \), \( \Omega_1(P) = \Omega_1(Z(P)) = \Omega_1(Z(H)) \triangleleft G \). Thus \( G = \langle P^G \rangle = C_G(\Omega_1(Z(P))) = C_G(\Omega_1(Z(H))) \). Let \( Q \in \text{Syl}_q(G) \), where prime \( q \neq p \) and \( Q \neq 1 \). Then \( Q \) acts trivially on \( \Omega_1(Z(H)) \) and thus \( Q \) acts trivially on \( Z(H) \) by Theorem 5.2.4 of [5]. Hence we have \( Q \leq C_G(Z(H)) \leq C_G(Z(P)) = P \). This is a contradiction and the theorem is proved.

**COROLLARY 2.** Let \( G \) be a finite group and \( P \in \text{Syl}_p(G) \). If \( \Omega_1(P) \leq Z(P) \) and \( N_G(Z(P)) \in N_p \), then \( G \in N_p \).

**PROOF.** Since \( Z(P) \) char \( P \), \( N_G(P), C_G(Z(P)) \leq N_G(Z(P)) \), and so \( N_G(P), C_G(Z(P)) \in N_p \). Then \( G \in N_p \) by Theorem 1.

**COROLLARY 3.** Let \( G \) be a finite group with a maximal subgroup \( M \) which is nilpotent and \( P \in \text{Syl}_2(M) \). If \( \Omega_2(P) \leq Z(P) \), then \( G \) is solvable.

**PROOF.** First of all we prove a lemma.

**LEMMA 1.** If \( P \) is a \( p \)-group with \( \Omega_2(P) \leq Z(P) \), then
\[
\Omega_2(P/Z(P)) \leq Z(P/Z(P)).
\]

**PROOF OF THE LEMMA.** In fact, we should prove \( [x, P] \leq Z(P) \) for each \( x \in P \) such that \( x^4 \in Z(P) \). Set \( P_i = P, P_{i+1} = [P, P_i], i = 1, 2, \ldots \). We use induction on \( |x| \) and inverse induction on \( i \). By the induction assumption, we have \( [x^2, P] \leq Z(P) \) and \( [x, P_{i+1}] \leq Z(P) \). Then for each \( y \in P_i \), we have
\[
1 = [x^4, y] = [x, y]^4[x, y, x, x]^2,
1 = [x^2, y, x] = [[x, y]^2[x, y, x], x] = [x, y, x]^2,
\]
thus \( [x, y]^4 = 1 \) and so \( [x, y] \in Z(P) \), that is \( [x, P_i] \leq Z(P) \), \( i = 1, 2, \ldots \). Especially \( [x, P] \leq Z(P) \). The lemma is proved.

Now we prove Corollary 3. Clearly \( N_G(Z(P)) \geq M \) by the nilpotency of \( M \). If \( N_G(Z(P)) = M \), then \( P \in \text{Syl}_2(G) \), for otherwise \( M < N_G(P) \leq N_G(Z(P)) \). However \( \Omega_1(P) \leq \Omega_2(P) \leq Z(P) \). Thus from Corollary 2, \( G \in N_2 \) and \( G \) is solvable by the odd order theorem \([4]\).

If \( N_G(Z(P)) > M \), \( Z(P) \triangleleft G \). Then from Lemma 1, \( G/Z(P) \) satisfies the condition of the corollary. We conclude \( G/Z(P) \) is solvable by induction on \( |G| \) and so is \( G \). This completes the proof of the corollary.

Clearly our Corollary 3 generalizes Thompson’s solvability theorem \([9]\). There are many other generalizations. In this paper, we mention two of them. In \([1]\), it is proved that if a solvable group \( A \) acts on a finite group \( G \) which has a nilpotent maximal \( A \)-invariant subgroup \( M \) with an abelian Sylow 2-subgroup, then \( G \) is solvable. In \([2]\), it is proved that if a finite group \( G \) has a nilpotent maximal subgroup \( M \) which has a Sylow 2-subgroup \( P \) with the property that each noncyclic subgroup of \( P \) containing \( Z(P) \) is normal in \( P \), then \( G \) is solvable. The latter result implies that if \( G \) has a nilpotent maximal subgroup \( M \) with a Sylow 2-subgroup \( P \), the class of which does not exceed 2, then \( G \) is solvable. This is the earlier theorem of Deskins-Janko \([6]\). We can unify these results. For this purpose, we first prove the following lemma.
Lemma 2. Let \( P \in \text{Syl}_2(G) \). If \( N_G(P_1)/C_G(P_1) \) is a 2-group for each noncyclic subgroup \( P_1 \) of \( P \) containing \( Z(P) \), then \( G \in N_2 \).

Proof. Let \( P_1 \) be a nonidentity subgroup of \( P \) containing \( Z(P) \). If \( P_1 \) is cyclic, \( N_G(P_1)/C_G(P_1) \) is a 2-group. If \( P_1 \) is not cyclic, \( N_G(P_1)/C_G(P_1) \) is also a 2-group by the hypothesis. Thus \( G \in N_2 \) by Theorem 3 of [3].

Theorem 4. Let \( G \) be a finite group on which a solvable group \( A \) acts. If \( G \) has a nilpotent maximal \( A \)-invariant subgroup \( M \) which has a Sylow 2-subgroup \( P \) with the property that each noncyclic subgroup \( P_1 \) of \( P \) containing \( Z(P) \) is \( A \)-invariant and normal in \( P \), then \( G \) is solvable.

Proof. If \( P < G \), then \( M/P \) is a maximal \( A \)-invariant subgroup of \( G/P \). Thus \( G/P \) is solvable by the theorem of [1] and so is \( G \). If \( M \) has a Sylow subgroup \( S \) of odd order which is normal in \( G \), then \( G/S \) is solvable by induction and so is \( G \). If each Sylow subgroup of \( M \) is not normal in \( G \), then \( M \) is a Hall subgroup of \( G \).

If each noncyclic subgroup \( P_1 \) of \( P \) containing \( Z(P) \) is not normal in \( G \), then \( N_G(P_1) = M \in N_2 \) and thus \( G \in N_2 \) by Lemma 2. So \( G \) is solvable by the odd order theorem. If there is a noncyclic subgroup of \( P \) containing \( Z(P) \) and normal in \( G \), we assume \( L \) is the maximal one of these subgroups and consider \( G/L \). \( M/L \) is a nilpotent maximal \( A \)-invariant subgroup of \( G/L \). The definition of \( L \) implies that for each nonidentity subgroup \( P_1/L \) of \( P/L \), \( P_1 \) is noncyclic and contains \( Z(P) \). By the hypothesis \( P_1 \) is \( A \)-invariant and normal in \( P \) and so \( P_1/L \) is \( A \)-invariant normal in \( P/L \). Thus \( N_{G/L}(P_1/L) = M/L \in N_2 \) and \( G/L \in N_2 \) by Lemma 2. Thus \( G/L \) is solvable and so is \( G \). This proves the theorem.

Theorem 5. Assume that a finite group \( A \) acts on a finite group \( G \) and \( G \) has a maximal \( A \)-invariant subgroup \( M \) which is nilpotent. Let \( P \in \text{Syl}_2(M) \). If each noncyclic subgroup \( P_1 \) of \( P \) which contains \( Z(P) \) is \( A \)-invariant and normal in \( P \) and one of the following conditions is satisfied, then \( G \) is solvable.

1. \( G \) has an \( A \)-invariant Sylow \( q \)-subgroup \( Q \neq 1 \) for some prime \( q \in \pi(G) - \pi(M) \).
2. \(|A|, |G:M| = 1\).

Proof. We first suppose that condition (1) is satisfied. If \( M = 1 \), then \( A \) acts irreducibly on \( G \). Thus \( G = Q \) and \( G \) is certainly solvable.

If \( M \neq 1 \) and \( M \) has an \( A \)-invariant subgroup \( K \) normal in \( G \), then \( M/K \) is a maximal \( A \)-invariant subgroup of \( G/K \). Clearly we have

\[
Z(P)K/K \leq Z(PK/K) \quad \text{and} \quad PK/K \cong P/P \cap K.
\]

This implies that \( P_1 \) is noncyclic and contains \( Z(P) \) for each noncyclic subgroup \( P_1K/K \) of \( PK/K \) containing \( Z(PK/K) \). Thus \( P_1 \) is \( A \)-invariant and normal in \( P \) by hypothesis. So \( P_1K/K \) is \( A \)-invariant and normal in \( PK/K \); \( G/K \) is solvable by induction. Then \( G \) is solvable.

Now we suppose that \( M \) has no \( A \)-invariant subgroup which is normal in \( G \). From the proof of Theorem 4, \( M \) is a Hall subgroup of \( G \).

We shall show that \( G \in N_p \) for each prime of \( \pi(M) \). Let \( S \in \text{Syl}_p(M) \). Clearly \( N_G(ZJ(S)) \) is \( A \)-invariant and so \( N_G(ZJ(S)) = M \in N_p \) by the preceding assumption. If \( p \) is an odd prime, then \( G \in N_p \) by Theorem 8.3.1 of [5]. If \( p = 2 \), since for
each noncyclic subgroup $P_i$ of $P$ containing $Z(P)$, $P_i$ is $A$-invariant and normal in $P$, we have $N_G(P_i) = M \in N_2$ by assumption. Then $G \in N_2$ by Lemma 2.

Let $R_p$ be the normal $p$-complement of $G$ for $p \in \pi(M)$. Let $R = \bigcap_{p \in \pi(M)} R_p$. Then $R$ is a normal complement of $G$ in $G$. So by the Frattini argument $G = R N_G(Q)$, where $Q$ is the $A$-invariant Sylow $q$-subgroup satisfying condition (1). We have that

$$R \cap N_G(Q) \triangleleft N_G(Q)$$

and

$$|R \cap N_G(Q)| = |R| |N_G(Q)|/|G| = |N_G(Q)|/|M|.$$  

Hence $R \cap N_G(Q)$ is a normal Hall subgroup of $N_G(Q)$. By Theorem 6.2.1 of [5], $R \cap N_G(Q)$ has a complement $M'$ in $N_G(Q)$ and clearly $|M'| = |M|$. So $M'$ is also a complement of $R$ in $G$, and $M' = M^{x'} \leq N_G(Q)$ for some $x' \in G$. It follows that $M \leq N_G(Q^x)$, where $x = x'^{-1}$.

We shall show that $Q^x$ is an $A$-invariant group. We first have $N_G(Q^{x'a}) \geq M$ for each $a \in A$ by the $A$-invariance of $M$. But $Q^{xa} = Q^{x'y}$ for some $y \in G$ by Sylow's theorem. So $N_G(Q^{xa}) = N_G(Q^{x'a})$. But then $M^{yu} \leq N_G(Q^{x'y}) = N_G(Q^{x'a})$ and, by a theorem of Weilandt [10], $M^{yu} = M$ for some $z \in N_G(Q^{x'a})$. Thus $yz \in N_G(M)$. If $N_G(M) > M$, then $M < G$ by the maximality of $M$. Thus $A$ acts irreducibly on $G/M$ and we have $G/M = QM/M$. Certainly $G$ is solvable. Now we assume $N_G(M) = M$; then $yz \in M \leq N_G(Q^{x'a})$ and so $y \in N_G(Q^{x'a})$. Thus $Q^x = Q^{x'a}$. This shows $Q^x$ is an $A$-invariant and so is $N_G(Q^x)$. Since $q \notin \pi(M)$, $N_G(Q^x) = G$ by the maximality of $M$. This shows $Q < G$. Thus $G = MQ$ and $G$ is solvable.

Suppose condition (2) is satisfied. Then $A$ acts on an $|A|^r$-group $R$. If $M = 1$, $A$ acts on an $|A|^r$-group $G$. By Theorem 7.7 of [7], $G$ is an elementary abelian group. If $m \neq 1$, $G$ has an $A$-invariant Sylow $q$-subgroup $Q$ contained in $R$ by Theorem 7.6 of [7], where $q \notin \pi(M)$. Thus condition (1) is satisfied and so $G$ is solvable. This completes the proof of the theorem.

If we put $A = 1$, then Theorems 4 and 5 will reduce to the result of [2]. If we set $P$ as an abelian group, then Theorem 4 will reduce to the result of [1].

Acknowledgment. The author would like to thank Professor Chen Zhongmu for his helpful advice.

References


Department of Mathematics, Southwest Normal University, Chongqing, China