PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 4, December 1986

ON TWO THEOREMS OF THOMPSON

ZHANG GUANGXIANG

ABSTRACT. Let G be a finite group.

THEOREM. Let $P \in \mathrm{Syl}_p(G)$ with $\Omega_1(P) \leq Z(P)$. If $N_G(Z(P))$ has a normal p-complement, then so does G.

COROLLARY. *Let M be a nilpotent maximal subgroup of G and $P \in \mathrm{Syl}_2(M)$ with $\Omega_2(P) \leq Z(P)$. Then G is solvable.*

This extends Thompson's solvability theorem [**9**]. We also give two other results generalizing Thompson's theorem.

In this note, we prove some theorems, one of which is similar to Thompson's normal p-complement theorem [**8**] and the others are generalizations of Thompson's theorem regarding solvability of finite groups [**9**].

For simplicity we write $X \in N_P$, which means the finite group X has a normal p-complement for a prime p. Our other notations are standard and follow [**5**].

THEOREM 1. *Let G be a finite group and $P \in \mathrm{Syl}_p(G)$. If $\Omega_1(P) \leq Z(P)$ and $N_G(P)$, $C_G(Z(P)) \in N_P$, then $G \in N_P$.*

PROOF. Let G be a counterexample of minimal order of the theorem. Then we deduce a contradiction step-by-step.

1. $O_{p'}(G) = 1$ by the minimality of $|G|$.

2. $O_p(G) \neq 1$, since otherwise we would have $N_G(P_1) < G$ for each subgroup of P_1 of P and $N_G(P_1) \in N_p$. In fact, we have a series of subgroups

$$N_G(P_1), N_G(P_2), \ldots, N_G(P_i), \ldots, N_G(P),$$

where $P_{i+1} \in \mathrm{Syl}_p(N_G(P_i))$. Now suppose $N_G(P_1) \notin N_p$ but $N_G(P) \in N_p$. There is some i_0 such that $N_G(P_{i_o}) \notin N_p$ while $N_G(P_{i_0+1}) \in N_p$. Then we have

$$\begin{aligned} &\Omega_1(P_{i_0+1}) \leq \Omega_1(P) \leq Z(P) \leq Z(P_{i_0+1}), \\ &C_G(Z(P_{i_0+1})) \leq C_G(Z(P)) \in N_p. \end{aligned} \qquad P_{i_0+1} \in \mathrm{Syl}_p(N_G(P_{i_0})),$$

Thus $N_G(P_{i_0}) \in N_p$ by the minimality of $|G|$. This is a contradiction. So $N_G(P_1) \in N_p$ for each $P_1 \leq P$ and hence $G \in N_p$ by Frobenius' theorem [**5**, Theorem 7.45]. This contradicts our hypothesis of G.

3. Set $H = O_p(G)$. Then $G/H \in N_p$ by a discussion similar to step 2. Let K/H be the normal p-complement of G/H. Then $1 \lhd H \lhd K \lhd G$ and G is p-solvable. Since $O_{p'}(G) = 1$, $C_G(H) \leq H$ by Theorem 6.3.2 of [**5**]. Particularly $Z(P) \leq H$.

4. Let M be a maximal subgroup of G containing P. Then $M \in N_p$ and $O_{p'}(M)H = O_{p'}(M) \times H$. This shows $O_{p'}(M) \leq C_G(H) \leq H$ and so $O_{p'}(M) = 1$.

Received by the editors October 25, 1985. Presented at the International Symposium on Group Theory in Beijing in 1984 sponsored by the Ministry of Education of the People's Republic of China.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20D20.

This forces $P = M$. Now we conclude $C_G(Z(P)) = P$, for otherwise $C_G(Z(P)) = G \in N_p$.

5. Since $\Omega_1(P) \leq Z(P) \leq Z(H)$, $\Omega_1(P) = \Omega_1(Z(P)) = \Omega_1(Z(H)) \lhd G$. Thus $G = \langle P^G \rangle = C_G(\Omega_1(Z(P))) = C_G(\Omega_1(Z(H)))$. Let $Q \in \mathrm{Syl}_q(G)$, where prime $q \neq p$ and $Q \neq 1$. Then Q acts trivially on $\Omega_1(Z(H))$ and thus Q acts trivially on $Z(H)$ by Theorem 5.2.4 of [**5**]. Hence we have $Q \leq C_G(Z(H)) \leq C_G(Z(P)) = P$. This is a contradiction and the theorem is proved.

COROLLARY 2. *Let G be a finite group and $P \in \mathrm{Syl}_p(G)$. If $\Omega_1(P) \leq Z(P)$ and $N_G(Z(P)) \in N_p$, then $G \in N_p$.*

PROOF. Since $Z(P)\,\mathrm{char}\,P$, $N_G(P), C_G(Z(P)) \leq N_G(Z(P))$, and so $N_G(P)$, $C_G(Z(P)) \in N_p$. Then $G \in N_p$ by Theorem 1.

COROLLARY 3. *Let G be a finite group with a maximal subgroup M which is nilpotent and $P \in \mathrm{Syl}_2(M)$. If $\Omega_2(P) \leq Z(P)$, then G is solvable.*

PROOF. First of all we prove a lemma.

LEMMA 1. *If P is a p-group with $\Omega_2(P) \leq Z(P)$, then*

$$\Omega_2(P/Z(P)) \leq Z(P/Z(P)).$$

PROOF OF THE LEMMA. In fact, we should prove $[x, P] \leq Z(P)$ for each $x \in P$ such that $x^4 \in Z(P)$. Set $P_1 = P$, $P_{i+1} = [P, P_i]$, $i = 1, 2, \ldots$. We use induction on $|x|$ and inverse induction on i. By the induction assumption , we have $[x^2, P] \leq Z(P)$ and $[x, P_{i+1}] \leq Z(P)$. Then for each $y \in P_i$, we have

$$1 = [x^4, y] = [x, y]^4 [x, y, x]^2,$$
$$1 = [x^2, y, x] = [[x, y]^2 [x, y, x], x] = [[x, y]^2, x] = [x, y, x]^2.$$

thus $[x, y]^4 = 1$ and so $[x, y] \in Z(P)$, that is $[x, P_i] \leq Z(P)$, $i = 1, 2, \ldots$. Especially $[x, P] \leq Z(P)$. The lemma is proved.

Now we prove Corollary 3. Clearly $N_G(Z(P)) \geq M$ by the nilpotency of M. If $N_G(Z(P)) = M$, then $P \in \mathrm{Syl}_2(G)$, for otherwise $M < N_G(P) \leq N_G(Z(P))$. However $\Omega_1(P) \leq \Omega_2(P) \leq Z(P)$. Thus from Corollary 2, $G \in N_2$ and G is solvable by the odd order theorem [**4**].

If $N_G(Z(P)) > M$, $Z(P) \lhd G$. Then from Lemma 1, $G/Z(P)$ satisfies the condition of the corollary. We conclude $G/Z(P)$ is solvable by induction on $|G|$ and so is G. This completes the proof of the corollary.

Clearly our Corollary 3 generalizes Thompson's solvability theorem [**9**]. There are many other generalizations. In this paper, we mention two of them. In [**1**], it is proved that if a solvable group A acts on a finite group G which has a nilpotent maximal A-invariant subgroup M with an abelian Sylow 2-subgroup, then G is solvable. In [**2**], it is proved that if a finite group G has a nilpotent maximal subgroup M which has a Sylow 2-subgroup P with the property that each noncyclic subgroup of P containing $Z(P)$ is normal in P, then G is solvable. The latter result implies that if G has a nilpotent maximal subgroup M with a Sylow 2-subgroup P, the class of which does not exceed 2, then G is solvable. This is the earlier theorem of Deskins-Janko [**6**]. We can unify these results. For this purpose, we first prove the following lemma.

LEMMA 2. *Let $P \in \mathrm{Syl}_2(G)$. If $N_G(P_1)/C_G(P_1)$ is a 2-group for each noncyclic subgroup P_1 of P containing $Z(P)$, then $G \in N_2$.*

PROOF. Let P_1 be a nonidentity subgroup of P containing $Z(P)$. If P_1 is cyclic, $N_G(P_1)/C_G(P_1)$ is a 2-group. If P_1 is not cyclic, $N_G(P_1)/C_G(P_1)$ is also a 2-group by the hypothesis. Thus $G \in N_2$ by Theorem 3 of [**3**].

THEOREM 4. *Let G be a finite group on which a solvable group A acts. If G has a nilpotent maximal A-invariant subgroup M which has a Sylow 2-subgroup P with the property that each noncyclic subgroup P_1 of P containing $Z(P)$ is A-invariant and normal in P, then G is solvable.*

PROOF. If $P \lhd G$, then M/P is a maximal A-invariant subgroup of G/P. Thus G/P is solvable by the theorem of [**1**] and so is G. If M has a Sylow subgroup S of odd order which is normal in G, then G/S is solvable by induction and so is G. If each Sylow subgroup of M is not normal in G, then M is a Hall subgroup of G.

If each noncyclic subgroup P_1 of P containing $Z(P)$ is not normal in G, then $N_G(P_1) = M \in N_2$ and thus $G \in N_2$ by Lemma 2. So G is solvable by the odd order theorem. If there is a noncyclic subgroup of P containing $Z(P)$ and normal in G, we assume L is the maximal one of these subgroups and consider G/L. M/L is a nilpotent maximal A-invariant subgroup G/L. The definition of L implies that for each nonidentity subgroup P_1/L of P/L, P_1 is noncyclic and contains $Z(P)$. By the hypothesis P_1 is A-invariant and normal in P and so P_1/L is A-invariant and normal in P/L. Thus $N_{G/L}(P_1/L) = M/L \in N_2$ and $G/L \in N_2$ by Lemma 2. Thus G/L is solvable and so is G. This proves the theorem.

THEOREM 5. *Assume that a finite group A acts on a finite group G and G has a maximal A-invariant subgroup M which is nilpotent. Let $P \in \mathrm{Syl}_2(M)$. If each noncyclic subgroup P_1 of P which contains $Z(P)$ is A-invariant and normal in P and one of the following conditions is satisfied, then G is solvable.*

(1) *G has an A-invariant Sylow q-subgroup $Q \neq 1$ for some prime $q \in \pi(G) - \pi(M)$.*

(2) $(|A|, |G : M|) = 1$.

PROOF. We first suppose that condition (1) is satisfied. If $M = 1$, then A acts irreducibly on G. Thus $G = Q$ and G is certainly solvable.

If $M \neq 1$ and M has an A-invariant subgroup K normal in G, then M/K is a maximal A-invariant subgroup of G/K. Clearly we have

$$Z(P)K/K \leq Z(PK/K) \quad \text{and} \quad PK/K \simeq P/P \cap K.$$

This implies that P_1 is noncyclic and contains $Z(P)$ for each noncyclic subgroup P_1K/K of PK/K containing $Z(PK/K)$. Thus P_1 is A-invariant and normal in P by hypothesis. So P_1K/K is A-invariant and normal in PK/K; G/K is solvable by induction. Then G is solvable.

Now we suppose that M has no A-invariant subgroup which is normal in G. From the proof of Theorem 4, M is a Hall subgroup of G.

We shall show that $G \in N_p$ for each prime of $\pi(M)$. Let $S \in \mathrm{Syl}_p(M)$. Clearly $N_G(ZJ(S))$ is A-invariant and so $N_G(ZJ(S)) = M \in N_p$ by the preceding assumption. If p is an odd prime, then $G \in N_p$ by Theorem 8.3.1 of [**5**]. If $p = 2$, since for

each noncyclic subgroup P_1 of P containing $Z(P)$, P_1 is A-invariant and normal in P, we have $N_G(P_1) = M \in N_2$ by assumption. Then $G \in N_2$ by Lemma 2.

Let R_p be the normal p-complement of G for $p \in \pi(M)$. Let $R = \bigcap_{p\in\pi(M)} R_p$. Then R is a normal complement of M in G. So by the Frattini argument $G = RN_G(Q)$, where Q is the A-invariant Sylow q-subgroup satisfying condition (1). We have that

$$R \cap N_G(Q) \lhd N_G(Q)$$

and

$$|R \cap N_G(Q)| = |R|\,|N_G(Q)|/|G| = |N_G(Q)|/|M|.$$

Hence $R \cap N_G(Q)$ is a normal Hall subgroup of $N_G(Q)$. By Theorem 6.2.1 of [**5**], $R \cap N_G(Q)$ has a complement M' in $N_G(Q)$ and clearly $|M'| = |M|$. So M' is also a complement of R in G, and $M' = M^{x'} \leq N_G(Q)$ for some $x' \in G$. It follows that $M \leq N_G(Q^x)$, where $x = x'^{-1}$.

We shall show that Q^x is an A-invariant group. We first have $N_G(Q^{xa}) \geq M$ for each $a \in A$ by the A-invariance of M. But $Q^{xa} = Q^{xy}$ for some $y \in G$ by Sylow's theorem. So $N_G(Q^x)^y = N_G(Q^{xa})$. But then $M^y \leq N_G(Q^{xy}) = N_G(Q^{xa})$ and, by a theorem of Weilandt [**10**], $M^{yz} = M$ for some $z \in N_G(Q^{xa})$. Thus $yz \in N_G(M)$. If $N_G(M) > M$, then $M < G$ by the maximality of M. Thus A acts irreducibly on G/M and we have $G/M = QM/M$. Certainly G is solvable. Now we assume $N_G(M) = M$; then $yz \in M \leq N_G(Q^{xa})$ and so $y \in N_G(Q^{xa})$. Thus $Q^x = Q^{xa}$. This shows Q^x is an A-invariant and so is $N_G(Q^x)$. Since $q \notin \pi(M)$, $N_G(Q^x) = G$ by the maximality of M. This shows $Q \lhd G$. Thus $G = MQ$ and G is solvable.

Suppose condition (2) is satisfied. Then A acts on an $|A|'$-group R. If $M = 1$, A acts on an $|A|'$-group G. By Theorem 7.7 of [**7**], G is an elementary abelian group. If $m \neq 1$, G has an A-invariant Sylow q-subgroup Q contained in R by Theorem 7.6 of [**7**], where $q \notin \pi(M)$. Thus condition (1) is satisfied and so G is solvable. This completes the proof of the theorem.

If we put $A = 1$, then Theorems 4 and 5 will reduce to the result of [**2**]. If we set P as an abelian group, then Theorem 4 will reduce to the result of [**1**].

ACKNOWLEDGMENT. The author would like to thank Professor Chen Zhongmu for his helpful advice.

REFERENCES

1. K. Brown, *Finite groups with nilpotent maximal A-invariant subgroups*, Arch. Math. (Basel) **22** (1971), 146–149.
2. Chen Miao Sheng, *A note on Glauberman theorem*, Tamkang J. Math. **7** (1976), 111–114.
3. Chen Zhongmu, *An extension of Burnside theorem*, Acta Math. Sinica **14** (1964), 75–77.
4. W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775–1029.
5. D. Gorenstein, *Finite groups*, 2nd ed., Chelsea, New York, 1980.
6. Z. Janko, *Finite groups with a nilpotent maximal subgroup*, J. Austral. Math. Soc. **4** (1964), 449–451.
7. H. Kurzweil, *Endliche Gruppen*, Springer-Verlag, Berlin and New York, 1977.
8. J. G. Thompson, *Normal p-complements of finite groups*, J. Algebra **1** (1964), 43–46.
9. ——, *Finite groups with fixed-point-free automorphism of prime order*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 578–581.
10. H. Wielandt, *Zum Satz von Sylow*, Math. Z. **60** (1954), 407–408.

DEPARTMENT OF MATHEMATICS, SOUTHWEST NORMAL UNIVERSITY, CHONGQING, CHINA