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MORREY SPACE

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ABSTRACT. For $1 \leq p < \infty$, Ω an open and bounded subset of R^n, and a nonincreasing and nonnegative function φ defined in $(0, \rho_0]$, $\rho_0 = \operatorname{diam} \Omega$, we introduce the space $\mathcal{M}^p_{\varphi,0}(\Omega)$ of locally integrable functions satisfying

$$\inf_{c \in C} \left\{ \int_{B(x_0,\rho)\cap\Omega} |f(x) - c|^p \, dx \right\} \leq A|B(x_0,\rho)|\varphi^p(\rho)$$

for every $x_0 \in \Omega$ and $0 < \rho \leq \rho_0$, where $|B(x_0,\rho)|$ denotes the volume of the ball centered in x_0 and radius ρ. The constant $A > 0$ does not depend on $B(x_0,\rho)$.

(i) We list some results on the structure, regularity, and density properties of the space so defined.

(ii) $\mathcal{M}^p_{\varphi,0}$ is represented as the dual of an atomic space.

Given an open and bounded subset Ω of R^n, let $\rho_0 = \operatorname{diam} \Omega$, $0 \leq \lambda \leq n$, $1 \leq p < \infty$. We denote by $L^{p,\lambda}(\Omega)$ the space introduced by Morrey of locally integrable functions $f(x)$ for which there is a constant $A = A(f) > 0$, such that

$$\int_{B(x_0,\rho)\cap\Omega} |f(x)|^p \, dx \leq A\rho^\lambda \tag{1}$$

for every $x_0 \in \Omega$, and $0 < \rho \leq \rho_0$, where $B(x_0,\rho) = \{x \in R^n / |x - x_0| < \rho\}$.

More generally, given $k \geq 0$, we say that $f(x)$ belongs to $\mathcal{L}^{p,\lambda}_k(\Omega)$ if there is a constant $A > 0$ such that for every $x_0 \in \Omega$, and $0 < \rho \leq \rho_0$ we have

$$\inf_{P \in \mathcal{P}_k} \left\{ \int_{B(x_0,\rho)\cap\Omega} |f(x) - P(x)|^p \, dx \right\} \leq A\rho^\lambda, \tag{2}$$

where $\mathcal{P}_k$ is the class of polynomials of degree $\leq k$.

We observe that $L^{p,0}(\Omega)$ describes $L^p(\Omega)$ and that $\mathcal{L}^{p,n}_0(\Omega)$ is a slightly more restricted version of BMO, the space of functions of bounded mean oscillation.

More generally, we can replace the second member in (1) by $|B(x_0,\rho)|\varphi^p(\rho)$, where $|\ |$ denotes the Lebesgue measure and φ is a function from $(0,\rho_0]$ into $[0,\infty)$. Let $M^p_\varphi(\Omega)$ be the space so defined. In the same way, using (2) we define the space $\mathcal{M}^p_{\varphi,k}(\Omega)$.

Condition (1) is obtained with $\varphi(t) = t^{(\lambda-n)/p}$. So the Morrey space is related to some nonincreasing function $\varphi(t)$ such that $\varphi(t) \to \infty$ when $t \to 0$. This is the kind of functions we will consider here.

Let us set

$$\|f\| = \|f\|_{L^p(\Omega)} + \inf_{A \in C} A^{1/p}.$$

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This is a norm in $\mathcal{M}^p_{\varphi,k}(\Omega)$ and in $M^p_\varphi(\Omega)$, so we get a Banach space.

Though R^n is not bounded, in the same manner we can define $M^p_\varphi(R^n)$ and $\mathcal{M}^p_{\varphi,k}(R^n)$. In this case we use M^p_φ and $\mathcal{M}^p_{\varphi,k}$.

We list some results on the structure, regularity, and density properties of the generalized Morrey space.

PROPOSITION 1. *Let Ω be an open and bounded subset of R^n. Suppose that there exists a constant $B > 0$ such that*

$$|\Omega \cap B(x_0, \rho)| \geq B\varphi^n \quad \textit{for all } x_0 \in \Omega,\ \rho \leq \rho_0.$$

Let $\varphi(t)$ satisfy the following conditions:

(i) *There is $0 < D < 1$ such that $\varphi(2t) \leq D\varphi(t)$ for $0 < t \leq \rho_0$.*

(ii) *$\varphi(t)$ is nonincreasing and $t^n\varphi^p(t)$ is nondecreasing.*

Then, $M^p_\varphi(\Omega)$ and $\mathcal{M}^p_{\varphi,k}(\Omega)$ describe the same space.

PROOF. This result has been proved by Campanato [**2**] when $\varphi(t) = t^{(\lambda-n)/p}$, $0 < \lambda < n$. The same proof applies in the general situation with minor changes.

Condition $0 < D < 1$ imposed in (i) implies that $\varphi(t)$ cannot be a constant function. In fact, in that case (1) would mean that $f(x)$ belongs to $L^\infty(\Omega)$ which is not true for every function in BMO.

Functions in $M^p_\varphi(\Omega)$ can be trivially extended to R^n.

PROPOSITION 2. *We suppose that $t^n\varphi^p(t)$ is nondecreasing in $(0, \infty)$. Then, given $f(x) \in M^p_\varphi(\Omega)$, its extension $f^*(x)$ defined as zero outside Ω belongs to M^p_φ.*

PROOF. It is simple. For $x_0 \in R^n$ and $\rho > 0$ it suffices to consider several cases: whether x_0 belongs to Ω or not and $\rho \leq \rho_0$ or not.

This result is also true for the space $\mathcal{M}^p_{\varphi,k}$ and includes the case $\varphi(t) = t^{(\lambda-n)/p}$ for $\lambda \geq 0$.

We can deduce that the definition of BMO as $\mathcal{L}^{p,n}_0(\Omega)$ is more restricted than the definition given in [**3**]. In fact, function $\log t$ belongs to $\mathrm{BMO}(R^+)$ but its extension as zero for $t \leq 0$ does not belong to BMO.

From the estimate obtained by John and Nirenberg in [**3**] for the distribution function in BMO, we deduce that when $\lambda = n$ and $k = 0$, condition (2) is satisfied independently of the exponent p which we have used. This is no longer true for $0 < \lambda < n$, because as was proved in [**1**], we can get functions in $M^p_\varphi(\Omega)$ with a distribution function arbitrarily large.

Functions in $M^p_\varphi(\Omega)$ cannot be approximated by functions in $C^\infty(\Omega)$, nor even by continuous functions. In fact, we find a simple example in $L^{p,\lambda}(\Omega)$, $0 < \lambda < n$: for $x_0 \in \Omega$ and $\rho_1 \leq \rho_0$ such that $B(x_0, \rho_1) \subset \Omega$, if we define

$$f_{x_0}(x) = |x - x_0|^{(\lambda-n)/p}, \quad x \in \Omega,$$

then we obtain

$$\|f - h\| \geq 2^{-p-1}|S^{n-1}|$$

for any continuous function $h(x)$ in Ω, where $S^{n-1} = \{x \in R^n / |x| = 1\}$. To see this it suffices to find $0 < \rho \leq \rho_1$ such that

$$\int_{B(x_0,\rho)} |f_{x_0}(x) - h(x)|^p\,dx \geq 2^{-p-1}|S^{n-1}|\rho^\lambda.$$

If $M = \sup_{x \in B(x_0,\rho_1)} |h(x)|^p$, then

$$\int_{B(x_0,\rho)} |f_{x_0}(x) - h(x)|^p\,dx \geq 2^{-p} \int_{B(x_0,\rho)} |f_{x_0}(x)|^p\,dx - \int_{B(x_0,\rho)} |h(x)|^p\,dx$$
$$\geq 2^{-p}|S^{n-1}|\rho^\lambda - M|S^{n-1}|\rho^n = |S^{n-1}|\rho^\lambda(2^{-p} - M\rho^{n-\lambda}).$$

Then it suffices to take $0 < \rho \leq \rho_1$ such that $(2^{-p} - M\rho^{n-\lambda}) \geq 2^{-p-1}$.

The situation changes when we consider the following subset of M^p_φ:

$$\overline{M}^p_\varphi = \{f \in M^p_\varphi \text{ such that } \|f(x-y) - f(x)\| \to 0 \text{ when } y \to 0\}.$$

We have

PROPOSITION 3. *Let $\varphi(t)$ be nonincreasing such that $t^n\varphi^p(t)$ is nondecreasing. Let $\psi(x) \in C^\infty(R^n)$ be supported in $B(0,1)$, $\int \psi(x)\,dx = 1$, $0 \leq \psi(x) \leq 1$, and $\psi_j(x) = j^n\psi(jx)$. Then*

(i) *If $f(x) \in \overline{M}^p_\varphi$, $f * \psi_j(x) \to f(x)$ in the M^p_φ norm as $j \to \infty$.*

(ii) *If $f(x)$ can be approximated by functions in C^1_0, then $f(x) \in \overline{M}^p_\varphi$.*

PROOF. (i) for $\rho > 0$, $\varepsilon > 0$

$$\left[\int_{B(x_0,\rho)} |f * \psi_j(x) - f(x)|^p\,dx\right]^{1/p}$$
$$\leq \left[\int_{|y|\leq 1/j} \psi_j(y)\,dy \int_{B(x_0,\rho)} |f(x-y) - f(x)|^p\,dx\right]^{1/p}$$
$$\leq \varepsilon\|f\|\rho^{n/p}\varphi(\rho) \quad \text{if } j \text{ is large.}$$

(ii) Let $\rho > 0$, $x_0 \in R^n$, $\varepsilon > 0$. Let $g(x) \in C^1_0(R^n)$ such that $\|f - g\| \leq \varepsilon/3$. Then

$$(*) \quad \left[\int_{B(x_0,\rho)} |f(x-y) - f(x)|^p\,dx\right]^{1/p} \leq \left[\int_{B(x_0\rho)} |f(x-y) - g(x-y)|^p\,dx\right]^{1/p}$$
$$+ \left[\int_{B(x_0,\rho)} |g(x) - f(x)|^p\,dx\right]^{1/p}$$
$$+ \left[\int_{B(x_0,\rho)} |g(x-y) - g(x)|^p\,dx\right]^{1/p}$$
$$\leq c2\varepsilon/3\rho^{n/p}\varphi(\rho) + \left[\int_{B(x_0,\rho)} |g(x-y) - g(x)|^p\,dx\right]^{1/p}.$$

Let $d > 1$ such that $\operatorname{supp}(g) \subset B(0, d-1)$. If we take $|y| < 1$, then

$$\leq c2\varepsilon/3\rho^{n/p}\varphi(\rho) + |y|\,\|\nabla g\|_\infty |B(0,d)|^{1/p}$$
$$(*) \quad \leq c2\varepsilon/3\rho^{n/p}\varphi(\rho) + \frac{|y|}{\varphi(d)}\|\nabla g\|_\infty c' d^{n/p}\varphi(d).$$

For $\rho \geq d$, since $t^n \varphi^p(t)$ is nondecreasing, it suffices to take

$$|y| \leq \frac{\varphi(d)\varepsilon/3}{\|\nabla g\|_\infty c'}.$$

If $\rho < d$, since $\varphi(t)$ is nonincreasing, we get

$$(*) \qquad \begin{aligned} &\leq c2\varepsilon/3\rho^{n/p}\varphi(\rho) + |y|\,\|\nabla g\|_\infty \rho^{n/p} \\ &\leq c2\varepsilon/3\rho^{n/p}\varphi(\rho) + \frac{|y|}{\varphi(d)}\|\nabla g\|_\infty c'\rho^{n/p}\varphi(\rho) \end{aligned}$$

as in the other case.

Proposition 3 is also true if we replace R^n for any $\Omega \subset R^n$, open and bounded.

Description of $\mathcal{M}^p_{\varphi,0}$ as a dual space. Fefferman and Stein have characterized BMO as the dual of the Hardy space H^1. $\mathcal{M}^p_{\varphi,0}$ can also be viewed as the dual of an "atomic" space.

For $1 < p < \infty$, and $B(x_0,\rho)$, we define $A^{p,\varphi}_{B(x_0,\rho)}$ as the set of functions $a(x)$ such that

(i) $\operatorname{supp}(a) \subset B(x_0,\rho)$,
(ii) $\int a(x)\,dx = 0$,
(iii) $\|a\|_p \leq 1/|B(x_0,\rho)|^{1/q}\varphi(\rho)$, $1/p + 1/q = 1$.

Let $A^{p,\varphi} = \bigcup_{B(x_0,\rho)} A^{p,\varphi}_{B(x_0,\rho)}$. We define $H^{p,\varphi}$ as the set of functions $f(x)$ such that $f(x) = \sum_{i\geq 0} \lambda_i a_i(x)$ in the sense of distributions, where $\lambda_i \in R$, $a_i(x) \in A^{p,\varphi}$, and $\sum_{i\geq 0} |\lambda_i| < \infty$.

We define a norm in $H^{p,\varphi}$

$$\|f\|_{H^{p,\varphi}} = \inf\left(\sum_{i\geq 0} |\lambda_i|\right).$$

Infimum if taken over all atomic decompositions of f.

In this section we only consider real valued functions.

PROPOSITION 4. *Let $1 < p < \infty$, $1/p + 1/q = 1$. If $\varphi(t)$ is nonincreasing and $t^n\varphi^q(t)$ is nondecreasing, then $H^{p,\varphi}$ is a Banach space.*

PROOF. Clearly $\|\ \|_{H^{p,\varphi}}$ defines a norm in $H^{p,\varphi}$. We must only see that $H^{p,\varphi}$ is complete.

If $\{f_n\}$ is a Cauchy sequence, we can choose a subsequence $\{f_{n_k}\}$ such that $\|f_{n_k} - f_{n_{k-1}}\|_{H^{p,\varphi}} \leq 2^{-k}$. We define

$$f = f_{n_1} + \sum_{k\geq 2}(f_{n_k} - f_{n_{k-1}}).$$

Let $f_{n_k} - f_{n_{k-1}} = \sum_{i\geq 0} \lambda_i^k a_i^k$ be the atomic decomposition such that

$$\sum_{i\geq 0} |\lambda_i^k| \leq \|f_{n_k} - f_{n_{k-1}}\|_{H^{p,\varphi}} + 2^{-k}.$$

Then $\sum_{k\geq 0}\sum_{i\geq 0} |\lambda_i^k| < \infty$ and we have a decomposition for f, except the series converges in the $H^{p,\varphi}$ norm. But this convergence also takes place in the sense of distributions.

If ψ is a testing function supported in $B(x_1,\rho_1)$ and a is an atom, we have

$$\left|\int a(x)\psi(x)\,dx\right| \le \|a\|_p\|\psi\|_q \le \frac{1}{|B(x,\rho)|^{1/q}\varphi(\rho)}\left(\int_{B(x,\rho)\cap B(x_1,\rho_1)}|\psi(x)|^q\,dx\right)^{1/q}.$$

If $\rho \le \rho_1$, then $\varphi(\rho) \ge \varphi(\rho_1)$ and

$$\left|\int a(x)\psi(x)\,dx\right| \le \frac{1}{|B(x,\rho)|^{1/q}\varphi(\rho_1)}\|\psi\|_\infty|B(x,\rho)|^{1/q} = \frac{\|\psi\|_\infty}{\varphi(\rho_1)}.$$

If $\rho \ge \rho_1$, then $\rho_1^{n/q}\varphi(\rho_1) \le \rho^{n/q}\varphi(\rho)$ and

$$\left|\int a(x)\psi(x)\,dx\right| \le \frac{1}{|B(x,\rho_1)|^{1/q}\varphi(\rho_1)}\|\psi\|_\infty|B(x,\rho_1)|^{1/q} = \frac{\|\psi\|_\infty}{\varphi(\rho_1)}.$$

Then, the result follows for any $g(x) \in H^{p,\varphi}$.

Consequently, since f_n converges to $f \in H^{p,\varphi}$ Proposition 4 is proved.

Finally, we have the following duality result.

PROPOSITION 5. *Let $\varphi(t) \ge 0$, $1 < p < \infty$, and $1/p + 1/q = 1$. For any $L \in (H^{p,\varphi})^*$ there exists $g \in \mathcal{M}^q_{\varphi,0}$ such that if $h(x) \in H^{p,\varphi}$ we have*

$$L(h) = \int g(x)h(x)\,dx.$$

Moreover, if $f \in \mathcal{M}^q_{\varphi,0}$ and $h \in H^{p,\varphi}$, then $\int f(x)h(x)\,dx$ is an element of $(H^{p,\varphi})^*$.

PROOF. The last statement is simple. For $f \in \mathcal{M}^q_{\varphi,0}$ and a an atom of $A^{p,\varphi}_{B(x_0,\rho)}$ we have

$$\int a(x)f(x)\,dx = \int a(x)(f(x) - c(x_0,\rho))\,dx,$$

where $c(x_0,\rho)$ is the constant for which the infimum in (2) is attained. Then

$$\begin{aligned}\int a(x)f(x)\,dx &\le \|a\|_p\left(\int_{B(x_0,\rho)}|f(x)-c(x_0,\rho)|^q\,dx\right)^{1/q}\\ &\le \frac{1}{|B(x_0,\rho)|^{1/q}\varphi(\rho)}\|f\|\,|B(x_0,\rho)|^{1/q}\varphi(\rho) = \|f\|.\end{aligned}$$

For any $h \in H^{p,\varphi}$ the affirmation follows immediately.

To prove the other statement we see first of all that $(H^{p,\varphi})^* \subset L^q_{\mathrm{loc}}$.

Let $L \in (H^{p,\varphi})^*$. Let B_k be an increasing sequence of balls which cover R^n. Let T_k be the restriction operator from R^n to B_k. Then $L \circ T_k$ belongs to $(L^p_0(B_k))^*$, where $L^p_0(B_k)$ denotes the subspace of $L^p(B_k)$ of functions having mean value zero. In fact, if $f \in L^p_0(B_k)$, then

$$|L(f)| \le \|L\|_{(H^{p,\varphi})^*}\|f\|_{H^{p,\varphi}} \le \|L\|_{(H^{p,\varphi})^*}\|f\|_p|B(x_0,\rho)|^{1/q}\varphi(\rho).$$

Since $(L^p_0(B_k))^* = L^q(B_k)/C(B_k)$ ($C(B_k)$ is the space of the functions that are constant on B_k) there exists $g_k \in L^q(B_k)$ such that

$$L(f) = \int f(x)g_k(x)\,dx.$$

Since the B_k are increasing we have $T_k(g_{k+1}) = g_k$. This implies the existence of a function $g \in L^q_{\text{loc}}$.

Now, we must prove that if $g \in L^q_{\text{loc}}$ is in $(H^{p,\varphi})^*$, then g belongs to $\mathcal{M}^q_{\varphi,0}$. To see this, we use a constant $c(x_0,\rho)$ for which

$$|\{x \in B(x_0,\rho)/g(x) < c(x_0,\rho)\}| \le 1/2|B(x_0,\rho)|,$$
$$|\{x \in B(x_0,\rho)/g(x) > c(x_0,\rho)\}| \ge 1/2|B(x_0,\rho)|,$$

and suppose without loss of generality that

$$\int_{B(x_0,\rho)\cap\{g(x)>c(x_0,\rho)\}} |g(x) - c(x_0,\rho)|^q\,dx \ge \int_{B(x_0,\rho)\cap\{g(x)\le c(x_0,\rho)\}} |g(x) - c(x_0,\rho)|^q\,dx.$$

To simplify, we denote $A = B(x_0,\rho)\cap\{g(x) > c(x_0,\rho)\}$ and $B = B(x_0,\rho)\cap\{g(x) \le c(x_0,\rho)\}$.

We define an atom $a(x)$ supported in $B(x_0,\rho)$ so

$$a(x) = [g(x) - c(x_0,\rho)]^{q-1} \quad \text{for } x \text{ in } A, \qquad a(x) = C \quad \text{in } B(x_0,\rho)\backslash A,$$

where C is a constant chosen so that the mean value of $a(x)$ over $B(x_0,\rho)$ is zero. We have

$$\begin{aligned}\int_{B(x_0,\rho)} |g(x) - c(x_0,\rho)|^q\,dx &\le 2\int_A |g(x) - c(x_0,\rho)|^q\,dx \\ &= 2\int_A (g(x) - c(x_0,\rho))a(x)\,dx \\ &\le 2\int g(x)a(x)\,dx \le 2\|g\|_{(H^{p,\varphi})^*}\|a\|_{H^{p,\varphi}}.\end{aligned}$$

Now,

$$\begin{aligned}&\|a\|_{H^{p,\varphi}} \le \|a\|_p|B(x_0,\rho)|^{1/q}\varphi(\rho) \\ &\le |B(x_0,\rho)|\varphi(\rho)\left[\frac{1}{|B(x_0,\rho)|}\int |a(x)|^p\,dx\right]^{1/p} \\ &\le |B(x_0,\rho)|\varphi(\rho)\left[\frac{1}{|B(x_0,\rho)|}\int_A |g(x) - c(x_0,\rho)|^{p(q-1)}\,dx + \frac{1}{|B(x_0,\rho)|}\int_B C^p\,dx\right]^{1/p}.\end{aligned}$$

But

$$\begin{aligned}\frac{1}{|B(x_0,p)|}\int_B C^p\,dx &\le \left[|B|^{-1}\int_B C\,dx\right]^p \\ &= \left[|B|^{-1}\int_A |g(x) - c(x_0,\rho)|^{q-1}\,dx\right]^p \\ &\le |B|^{-1}\int_A |g(x) - c(x_0,\rho)|^q\,dx.\end{aligned}$$

Hence

$$\|a\|_{H^{p,\varphi}} \le |B(x_0,\rho)|\varphi(\rho)\left[\frac{1}{|B(x_0,\rho)|}\int_A |g(x) - c(x_0,\rho)|^q\,dx\right]^{1/p}.$$

Then

$$\left[\int_{B(x_0,\rho)} |g(x) - c(x_0,\rho)|^q \, dx\right]^{1/q} \leq c|B(x_0,\rho)|^{1/q}\varphi(\rho).$$

So, Proposition 5 is proved.

References

1. J. Alvarez Alonso, *The distribution function in the Morrey space*, Proc. Amer. Math. Soc. **83** (1981), 693–699.
2. S. Campanato, *Proprietà di una famiglia di spazi funzionali*, Ann. Sci. Norm. Sup. Pisa **18** (1964), 137–160.
3. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1964), 415–426.
4. J. L. Journé, *Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*, Springer-Verlag, 1983.
5. J. Peetre, *On the theory of $L^{p,\lambda}$ spaces*, J. Funct. Anal. **4** (1964), 71–87.

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