A LATTICE-THEORETIC EQUIVALENT OF THE INVARIANT SUBSPACE PROBLEM
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ABSTRACT. Every bounded linear operator on complex infinite-dimensional separable Hilbert space has a proper invariant subspace if and only if for every lattice automorphism \( \phi \) of a certain abstract complete lattice \( P \) (defined by N. Zierler) there exists an element \( a \in P \) different from 0 and 1 such that \( \phi^2(a) \leq a \).

Let \( H \) be a complex, infinite-dimensional, separable Hilbert space. The Invariant Subspace Problem is: does every operator \( T \in B(H) \) have a nontrivial invariant subspace? Several equivalent problems are known; some are mentioned in [3] (e.g. p. 190, p. 194), see also [2]. In this note we point out that this famous problem is equivalent to a problem in lattice theory (Corollary 1).

Our equivalence rests on two results. Firstly, the lattice \( C(H) \) of all closed subspaces of \( H \) has a lattice-theoretic characterization due to Zierler [4, 5]. Secondly, every lattice automorphism \( \phi \) of \( C(H) \) is spatial in the sense that there exists a bicontinuous linear or conjugate linear bijection (unique up to nonzero scalar multiples) \( S: H \to H \) such that \( \phi(M) = SM \) for every \( M \in C(H) \), [1].

Let \( \text{Aut } C(H) \) denote the group of automorphisms of \( C(H) \).

**THEOREM 1.** The following are equivalent.

1. For every \( T \in B(H) \) there exists \( M \in C(H) \) different from (0) and \( H \) such that \( TM \subseteq M \);
2. For every \( \phi \in \text{Aut } C(H) \) there exists \( M \in C(H) \) different from (0) and \( H \) such that \( \phi^2(M) \subseteq M \).

**PROOF.** Assume (1) holds. Let \( \phi \in \text{Aut } C(H) \) be induced by \( S \). Then \( \phi^2 \) is induced by \( S^2 \) which is linear. Thus (2) holds.

Conversely, assume (2) holds. Let \( T \in B(H) \) and let \( \lambda \) be a scalar satisfying \( |\lambda| > \|T\| \). Then \( S = T - \lambda \) is invertible and [3, p. 34] \( S = R^2 \) for some invertible operator \( R \in B(H) \). If \( \phi \) is the automorphism induced by \( R \) and \( M \) is as in (2) above, we have \( SM \subseteq M \) so \( TM \subseteq M \).

Let \( P \) be an abstract lattice satisfying the (lattice-theoretic) hypotheses of Theorem 2.2 of [4] and assume also that (in the notation of [4]) the coordinatizing division ring \( D \) is algebraically closed. Then there exists a lattice isomorphism \( \theta: P \to C(H) \) (satisfying \( \theta(a') = \theta(a)^\perp \), where \( a' \) denotes the complement of \( a \) in \( P \)). Let \( \text{Aut } P \) denote the set of automorphisms of \( P \), and let 0 (respectively, 1) denote the least (respectively, greatest) element of \( P \).
COROLLARY 1. The following are equivalent.

(1) For every $T \in B(H)$ there exists $M \in \mathcal{C}(H)$ different from $(0)$ and $H$ such that $TM \subseteq M$;

(2) For every $\phi \in \text{Aut } P$ there exists $a \in P$ different from 0 and 1 such that $\phi^2(a) \leq a$.

Let $\mathcal{L}$ denote the set of automorphisms of $\mathcal{C}(H)$ that are induced by invertible operators of $B(H)$. Obviously $\mathcal{L}$ is a subgroup of Aut $\mathcal{C}(H)$. Some characterizations of $\mathcal{L}$ follow.

THEOREM 2. $\mathcal{L} = \{\phi^2 \psi^2 : \phi, \psi \in \text{Aut } \mathcal{C}(H)\}$.

PROOF. Let $\mathcal{Q} = \{\phi^2 \psi^2 : \phi, \psi \in \text{Aut } \mathcal{C}(H)\}$. Clearly $\mathcal{Q} \subseteq \mathcal{L}$. Let $\eta \in \mathcal{L}$ and suppose that $\eta$ is induced by $S \in B(H)$. Then $S = UA$ with $U$ unitary and $A$ positive and invertible. There exists $V \in B(H)$ such that $V^2 = U$. Also $(A^{1/2})^2 = A$. Thus $\eta = \phi^2 \psi^2$ where $\phi$ is induced by $V$ and $\psi$ is induced by $A^{1/2}$.

COROLLARY 2. $\mathcal{L}$ is the subgroup of Aut $\mathcal{C}(H)$ generated by $\{\phi^2 : \phi \in \text{Aut } \mathcal{C}(H)\}$.

If $G$ is an abstract group, it is clear that a subgroup $K$ of $G$ has index 2 if and only if $gh \in K$ whenever $g, h \notin K$. Also, every subgroup of $G$ of index 2 is a maximal proper (normal) subgroup and contains the square of every element of $G$. From these observations and the preceding corollary it follows that $\mathcal{L}$ is the only subgroup of Aut $\mathcal{C}(H)$ of index 2.

REFERENCES


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