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EXTREME POINTS IN $C(K, L^{\phi}(\mu))$

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ABSTRACT. Let $L^{\phi}(\mu)$ denote an Orlicz space and let ϕ satisfy the condition Δ_2. It is shown that the extreme points of the unit ball of the space of continuous functions from a compact Hausdorff space K into $L^{\phi}(\mu)$ with supremum norm on $C(K, L^{\phi}(\mu))$ are precisely the functions with values in the set of extreme points of the unit ball of $L^{\phi}(\mu)$.

1. Introduction. Let K be a compact Hausdorff space and let X be a real Banach space. By $C(K, X)$ we denote the Banach space of all X-valued continuous functions on K equipped with the norm $\|f\| = \sup_{k \in K}\|f(k)\|$. The purpose of this paper is to study the extreme points of the unit ball of $C(K, L^{\phi}(\mu))$ ($L^{\phi}(\mu)$ denotes an Orlicz space). Our result is a generalization of Werner's result for $C(K, L^1(\mu))$ [**18**]. We denote by ext X the set of all extreme points of the unit ball of the Banach space X. Obviously, if $\{k \in K\colon f(x) \in \operatorname{ext} X\} = f^{-1}(\operatorname{ext} X)$ is dense in K, then $f \in \operatorname{ext} C(K, X)$. Thus in the case when ext X is closed, a natural conjecture is that $f \in \operatorname{ext} C(K, X)$ if and only if $f(k) \in \operatorname{ext} X$ for all $k \in K$. It is known that this conjecture is true in many cases (for instance if X is strictly convex), but it is not true in general. Blumenthal, Lindenstrauss and Phelps [**2**] have, actually, presented an example of a four-dimensional space X and function $f \in \operatorname{ext} C([0,1], X)$ such that $f(k) \notin \operatorname{ext} X$ for all $k \in [0,1]$.

In fact, the problem of which spaces X satisfy the above conjecture was considered by Clausing and Papadopoulou [**3**]. If the unit ball $B(X)$ of X is stable, then the correspondence $x \to D(x) = \{z \in B(X)\colon$ there exists $z_1 \in B(X)$ with $(z + z_1)/2 = x\}$ is lower semicontinuous. Thus by applying the Michael's selection theorem (cf. [**10**]) we can represent each function $f \in C(K, X)$ satisfying $f^{-1}(\operatorname{ext} X) \neq K$ ($\|f\| \leqslant 1$) as a nontrivial convex combination of norm-one elements (ext X is closed for stable ball). It turns out that if X is finite dimensional, then $B(X)$ is stable if and only if all n-skeletons $\{x \in B(X)\colon \dim \operatorname{face}(x) \leqslant n\}$ of $B(X)$ are closed, $n = 0, 1, \ldots, \dim X$ (see Papadopoulou [**12**]). Note that ext X = 0-skeleton of $B(X)$. Since ($\dim X - 1$)- and ($\dim X - 2$)-skeletons are always closed the conjecture is true for all X with $\dim X = 2$ or if ext X is closed in the case $\dim X = 3$. In [**7**] it was shown that the unit ball of a finite-dimensional Orlicz space is stable. Applying arguments from [**16**] we may see that Banach spaces X with 3.2.I.P. also

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satisfy the conjecture. The above conjecture was also considered in [**19**]. We should point out that $C(K, X^*)$ (X^* denotes the dual of X) is isometrically isomorphic to the space of all compact linear operators from X into $C(K)$ (see [**4**, p. 490]). Thus our conjecture is naturally connected with the problem of characterization of extreme operators in $\mathscr{K}(X, C(K))$ (note that the space of bounded linear operators $\mathscr{L}(X, C(K))$ is isometrically isomorphic to the space of weak*-continuous functions from K into X^*). In particular, the question is, whether an extreme operator $T \in \mathscr{K}(X, C(K))$ must be "nice" in the sense of Morris and Phelps [**11**]; i.e., whether T maps extreme functionals in $C(K)^*$ (Dirac measure on K) into extreme points of the unit ball of X^*. Many authors have considered this problem (see [**2, 1, 13, 5, 8**]; see also [**14, 15**] for negative results). Another negative example was given by Greim [**6**] for vector-valued L^p-functions.

2. Ext $C(K, L^\phi(\mu))$. Let (Ω, Σ, μ) be a σ-finite measure space. Let ϕ: $\mathbf{R} \to [0, \infty)$ be a convex, even function with $\phi(0) = 0$. We say that ϕ satisfies the condition Δ_2 if there exists a constant $M > 0$ such that the inequality $\phi(2t) \leqslant M\phi(t)$ holds for all t. In some particular cases the assumption on ϕ can be weakened: if μ is purely atomic with $\inf b_k > 0$ (where b_k denote masses of atoms) it suffices to assume that ϕ satisfies the condition Δ_2 for small values of t; if $\mu(\Omega) < \infty$ then we only need to assume that ϕ satisfies the condition Δ_2 for large values of t and $\phi(t) = 0$ exactly when $t = 0$.

The Orlicz space $L^\phi(\mu)$ is the set of (equivalence classes of) measurable functions x: $\Omega \to \mathbf{R}$ such that $\int \phi(\lambda x) < \infty$ for some $\lambda > 0$ equipped with the Luxemburg norm $\|x\|_\phi = \inf\{\alpha > 0\colon \int \phi(x/\alpha)\, d\mu \leqslant 1\}$. Note that if ϕ satisfies the condition Δ_2, then $\int \phi(x/\|x\|_\phi) = 1$ for all $0 \neq x \in L^\phi(\mu)$. We refer the reader to [**9**] for basis facts about Orlicz spaces.

Let ϕ satisfy the condition Δ_2. Let U_ϕ be a maximal open subset of $\mathbf{R}$ such that ϕ is linear on each connected component of U_ϕ. It is not difficult to see that $x \notin \operatorname{ext} L^\phi(\mu)$ if and only if $x^{-1}(U_\phi)$ has positive measure and is not an atom (i.e. can be divided into two subsets of positive measure). Therefore, in particular, $L^\phi(\mu)$ is strictly convex if $U_\phi = \varnothing$ (i.e. ϕ is strictly convex); cf. [**17**].

THEOREM. *Suppose ϕ satisfies the condition Δ_2. Then f is extremal in the unit ball of $C(K, L^\phi(\mu))$ if and only if $f(k)$ is extremal in the unit ball of $L^\phi(\mu)$ for all $k \in K$.*

PROOF. Suppose that $f(k_0) \notin \operatorname{ext} L^\phi(\mu)$ for some $k_0 \in K$. We may and do assume that $\|f(k)\|_\phi = 1$ for all $k \in K$. To prove our theorem we need to show that $f \notin \operatorname{ext} C(K, L^\phi(\mu))$. There exist norm one $x_1, x_2 \in L^\phi(\mu)$ such that $x_1 \neq x_2$ and $(x_1 + x_2)/2 = f(k_0)$. Put

$$A_1 = \left\{w\colon \left|[f(k_0)](w)\right| > |x_1(w)|\right\}, \qquad B_1 = \left\{w\colon \left|[f(k_0)](w)\right| < |x_1(w)|\right\}.$$

We have $\mu(A_1) > 0$ and $\mu(B_1) > 0$. It is easy to see that $[f(k_0)](A_1) \subset U_\phi$ and $[f(k_0)](B_1) \subset U_\phi$. Thus there exist $A \subset A_1$, $B \subset B_1$ of finite and positive measure such that $[f(k_0)](A)$ is contained in one connected component of U_ϕ and $[f(k_0)](B)$ is contained in one connected component of U_ϕ. Hence there exist a_1, a_2, b_1,

$b_2 \in \mathbf{R}$ such that $[f(k_0)](A) \subset (a_2, a_2) \subset U_\phi$ and $[f(k_0)](B) \subset (b_1, b_2) \subset U_\phi$. For $x \in L^\phi(\mu)$ we define $\eta_A(x)$, $\eta_B(x) \in L^\phi(\mu)$ by

$$[\eta_A(x)](w) = 1_A \max\{0, \min(a_2 - x(w)), (x(w) - a_1)\},$$
$$[\eta_B(x)](w) = 1_B \max\{0, \min(b_2 - x(w)), (x(w) - b_1)\}.$$

We have

$$\tfrac{1}{2}\{\phi([x + \eta_A(x)](w)) + \phi([x - \eta_A(x)](w))\} = \phi(x(w)),$$
$$\tfrac{1}{2}\{\phi([x + \eta_B(x)](x)) + \phi([x - \eta_B(x)](w))\} = \phi(x(w)),$$

and

$$(x \pm \eta_A(x))(w) \in [a_1, a_2] \quad \text{if } x(w) \in [a_1, a_2],$$
$$(x \pm \eta_B(x))(w) \in [b_1, b_2] \quad \text{if } x(w) \in [b_1, b_2].$$

Obviously, $\eta_A(f(k_0)) \neq 0 \neq \eta_B(f(k_0))$. We should add that functions $L^\phi(\mu) \ni x \to \eta_A(x) \in L^\phi(\mu)$ and $L^\phi(\mu) \ni x \to \eta_B(x) \in L^\phi(\mu)$ are continuous. Thus $\eta_A(f(\cdot))$, $\eta_B(f(\cdot)) \in C(K, L^\phi(\mu))$. Put

$$m_A = \phi'((a_1 + a_2)/2), \qquad m_B = \phi'((b_1 + b_2)/2).$$

We have

$$\phi(a_1 + t) = \phi(a_1) + m_A t \quad \text{for } t \in (0, a_2 - a_1)$$

and

$$\phi(b_1 + t) = \phi(b_1) + m_B t \quad \text{for } t \in (0, b_2 - b_1).$$

We define functions p_A, p_B: $L^\phi(\mu) \to \mathbf{R}$ by

$$p_A(x) = m_A \int \eta_A(x)\, d\mu, \qquad p_B(x) = m_B \int \eta_B(x)\, d\mu, \qquad x \in L^\phi(\mu).$$

The functions p_A, p_B are continuous. Indeed, first consider p_A. Let $\|x_n - x_0\|_\phi \to 0$, where $x_n \in L^\phi(\mu)$. Thus $\|\eta_A(x_n) - \eta_A(x_0)\|_\phi \to 0$. Since $\mu(A) < \infty$, there exists a constant $c > 0$ such that $\|1_A y\|_{L^1(\mu)} \leqslant c\|1_A y\|_\phi$ for all $y \in L^\phi(\mu)$. Therefore, we obtain $\|\eta_A(x_n) - \eta_A(x_0)\|_{L^1(\mu)} \to 0$. Hence $\int \eta_A(x_n)\, d\mu \to \int \eta_A(x_0)\, d\mu$, i.e. $p_A(x_n) \to p_A(x_0)$. Analogously we obtain that p_B is continuous. Thus $p_A(f(\cdot))$, $p_B(f(\cdot)) \in C(K, \mathbf{R})$.

Now we define $f_i \in C(K, L^\phi(\mu))$, $i = 1, 2$, by

$$f_i = f + (-1)^i [\alpha_A \eta_A(f) - \alpha_B \eta_B(f)]$$

where

$$\alpha_A = \frac{p_B(f)}{1 + |p_A(f)| + |p_B(f)|}, \qquad \alpha_B = \frac{p_A(f)}{1 + |p_A(f)| + |p_B(f)|}.$$

Obviously, α_A, $\alpha_B \in C(K, \mathbf{R})$, $|\alpha_A| \leqslant 1$, $|\alpha_B| \leqslant 1$.

We have $f = (f_1 + f_2)/2$ and $f_i(k_0) \neq f(k_0)$. We need to check that $\|f_i\| \leqslant 1$, $i = 1, 2$. We have

$$\int \phi(f_1(k))\,d\mu = \int_{\Omega\setminus(A\cup B)} \phi(f(k))\,d\mu + \int_A \phi(f(k)) - m_A\alpha_A\eta_A(f)\,d\mu$$

$$+ \int_B \phi(f(k)) + m_B\alpha_B\eta_B(f)\,d\mu$$

$$= \int_{\Omega\setminus A\cup B} \phi(f(k))\,d\mu + \int_{A\cup B} \phi(f(k))\,d\mu = \int \phi(f(k))\,d\mu = 1.$$

Thus $\|f_1(k)\| = 1$. Analogously $\|f_2(k)\| = 1$ for all $k \in K$. Therefore, $f \in \operatorname{ext} C(K, L^\phi(\mu))$.

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