EXTREME POINTS IN $C(K, L^\phi(\mu))$

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Abstract. Let $L^\phi(\mu)$ denote an Orlicz space and let $\phi$ satisfy the condition $\Delta_2$. It is shown that the extreme points of the unit ball of the space of continuous functions from a compact Hausdorff space $K$ into $L^\phi(\mu)$ with supremum norm on $C(K, L^\phi(\mu))$ are precisely the functions with values in the set of extreme points of the unit ball of $L^\phi(\mu)$.

1. Introduction. Let $K$ be a compact Hausdorff space and let $X$ be a real Banach space. By $C(K, X)$ we denote the Banach space of all $X$-valued continuous functions on $K$ equipped with the norm $\|f\| = \sup_{k \in K} \|f(k)\|$. The purpose of this paper is to study the extreme points of the unit ball of $C(K, L^\phi(\mu))$ ($L^\phi(\mu)$ denotes an Orlicz space). Our result is a generalization of Werner’s result for $C(K, L^1(\mu))$ [18]. We denote by $\text{ext} X$ the set of all extreme points of the unit ball of the Banach space $X$. Obviously, if \{ $k \in K$: $f(x) \in \text{ext} X$ $=$ $f^{-1}(\text{ext} X)$ is dense in $K$, then $f \in \text{ext} C(K, X)$. Thus in the case when $\text{ext} X$ is closed, a natural conjecture is that $f \in \text{ext} C(K, X)$ if and only if $f(k) \in \text{ext} X$ for all $k \in K$. It is known that this conjecture is true in many cases (for instance if $X$ is strictly convex), but it is not true in general. Blumenthal, Lindenstrauss and Phelps [2] have, actually, presented an example of a four-dimensional space $X$ and function $f \in \text{ext} C([0,1], X)$ such that $f(k) \not\in \text{ext} X$ for all $k \in [0,1]$.

In fact, the problem of which spaces $X$ satisfy the above conjecture was considered by Clausing and Papadopoulou [3]. If the unit ball $B(X)$ of $X$ is stable, then the correspondence $x \rightarrow D(x) = \{ z \in B(X):$ there exists $z_1 \in B(X)$ with $(z + z_1)/2 = x \}$ is lower semicontinuous. Thus by applying the Michael’s selection theorem (cf. [10]) we can represent each function $f \in C(K, X)$ satisfying $f^{-1}(\text{ext} X) \neq K (\|f\| \leq 1)$ as a nontrivial convex combination of norm-one elements (ext $X$ is closed for stable ball). It turns out that if $X$ is finite dimensional, then $B(X)$ is stable if and only if all $n$-skeletons $\{ x \in B(X): \dim \text{face}(x) \leq n \}$ of $B(X)$ are closed, $n = 0, 1, \ldots, \dim X$ (see Papadopoulou [12]). Note that $\text{ext} X = 0$-skeleton of $B(X)$. Since ($\dim X - 1$)- and ($\dim X - 2$)-skeletons are always closed the conjecture is true for all $X$ with $\dim X = 2$ or if ext $X$ is closed in the case $\dim X = 3$. In [7] it was shown that the unit ball of a finite-dimensional Orlicz space is stable. Applying arguments from [16] we may see that Banach spaces $X$ with 3.2.I.P. also
satisfy the conjecture. The above conjecture was also considered in [19]. We should point out that $C(K, X^*)$ ($X^*$ denotes the dual of $X$) is isometrically isomorphic to the space of all compact linear operators from $X$ into $C(K)$ (see [4, p. 490]). Thus our conjecture is naturally connected with the problem of characterization of extreme operators in $\mathcal{X}(X, C(K))$ (note that the space of bounded linear operators $\mathcal{L}(X, C(K))$ is isometrically isomorphic to the space of weak*-continuous functions from $K$ into $X^*$). In particular, the question is, whether an extreme operator $T \in \mathcal{X}(X, C(K))$ must be "nice" in the sense of Morris and Phelps [11]; i.e., whether $T$ maps extreme functionals in $C(K)^*$ (Dirac measure on $K$) into extreme points of the unit ball of $X^*$. Many authors have considered this problem (see [2, 1, 13, 5, 8]; see also [14, 15] for negative results). Another negative example was given by Greim [6] for vector-valued $L^p$-functions.

2. Ext $C(K, L^p(\mu))$. Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space. Let $\phi: \mathbb{R} \to [0, \infty)$ be a convex, even function with $\phi(0) = 0$. We say that $\phi$ satisfies the condition $\Delta_2$ if there exists a constant $M > 0$ such that the inequality $\phi(2t) \leq M\phi(t)$ holds for all $t$. In some particular cases the assumption on $\phi$ can be weakened: if $\mu$ is purely atomic with $\inf b_k > 0$ (where $b_k$ denote masses of atoms) it suffices to assume that $\phi$ satisfies the condition $\Delta_2$ for small values of $t$; if $\mu(\Omega) < \infty$ then we only need to assume that $\phi$ satisfies the condition $\Delta_2$ for large values of $t$ and $\phi(t) = 0$ exactly when $t = 0$.

The Orlicz space $L^\phi(\mu)$ is the set of (equivalence classes of) measurable functions $x: \Omega \to \mathbb{R}$ such that $\int \phi(\lambda x) < \infty$ for some $\lambda > 0$ equipped with the Luxemburg norm $\|x\|_\phi = \inf\{\alpha > 0: \int \phi(x/\alpha) d\mu < 1\}$. Note that if $\phi$ satisfies the condition $\Delta_2$, then $\int \phi(x/\|x\|_\phi) = 1$ for all $0 \neq x \in L^\phi(\mu)$. We refer the reader to [9] for basis facts about Orlicz spaces.

Let $\phi$ satisfy the condition $\Delta_2$. Let $U_\phi$ be a maximal open subset of $\mathbb{R}$ such that $\phi$ is linear on each connected component of $U_\phi$. It is not difficult to see that $x \notin \text{ext } L^\phi(\mu)$ if and only if $x^{-1}(U_\phi)$ has positive measure and is not an atom (i.e. can be divided into two subsets of positive measure). Therefore, in particular, $L^\phi(\mu)$ is strictly convex if $U_\phi = \emptyset$ (i.e. $\phi$ is strictly convex); cf. [17].

**Theorem.** Suppose $\phi$ satisfies the condition $\Delta_2$. Then $f$ is extremal in the unit ball of $C(K, L^\phi(\mu))$ if and only if $f(k)$ is extremal in the unit ball of $L^\phi(\mu)$ for all $k \in K$.

**Proof.** Suppose that $f(k_0) \notin \text{ext } L^\phi(\mu)$ for some $k_0 \in K$. We may and do assume that $\|f(k_0)\|_\phi = 1$ for all $k \in K$. To prove our theorem we need to show that $f \notin \text{ext } C(K, L^\phi(\mu))$. There exist norm one $x_1, x_2 \in L^\phi(\mu)$ such that $x_1 \neq x_2$ and $(x_1 + x_2)/2 = f(k_0)$. Put

$$A_1 = \{w: |[f(k_0)](w)| > |x_1(w)|\}, \quad B_1 = \{w: |[f(k_0)](w)| < |x_1(w)|\}.$$

We have $\mu(A_1) > 0$ and $\mu(B_1) > 0$. It is easy to see that $[f(k_0)](A_1) \subset U_\phi$ and $[f(k_0)](B_1) \subset U_\phi$. Thus there exist $A \subset A_1$, $B \subset B_1$ of finite and positive measure such that $[f(k_0)](A)$ is contained in one connected component of $U_\phi$ and $[f(k_0)](B)$ is contained in one connected component of $U_\phi$. Hence there exist $a_1, a_2, b_1,
\( b_2 \in \mathbb{R} \) such that \([f(k_0)](A) \subset (a_2, a_2) \subset U_\phi \) and \([f(k_0)](B) \subset (b_1, b_2) \subset U_\phi \). For \( x \in L^\phi(\mu) \) we define \( \eta_A(x), \eta_B(x) \in L^\phi(\mu) \) by
\[
[\eta_A(x)](w) = 1_A \max\{0, \min(a_2 - x(w), (x(w) - a_1)\},
[\eta_B(x)](w) = 1_B \max\{0, \min(b_2 - x(w), (x(w) - b_1)\}.
\]

We have
\[
\frac{1}{2} \{ \phi([x + \eta_A(x)](w)) + \phi([x - \eta_A(x)](w)) \} = \phi(x(w)),
\]
\[
\frac{1}{2} \{ \phi([x + \eta_B(x)](w)) + \phi([x - \eta_B(x)](w)) \} = \phi(x(w)),
\]
and
\[
(x \pm \eta_A(x))(w) \in [a_1, a_2] \quad \text{if} \ x(w) \in [a_1, a_2],
(x \pm \eta_B(x))(w) \in [b_1, b_2] \quad \text{if} \ x(w) \in [b_1, b_2].
\]

Obviously, \( \eta_A(f(k_0)) \neq 0 \neq \eta_B(f(k_0)) \). We should add that functions \( L^\phi(\mu) \ni x \to \eta_A(x) \in L^\phi(\mu) \) and \( L^\phi(\mu) \ni x \to \eta_B(x) \in L^\phi(\mu) \) are continuous. Thus \( \eta_A(f(\cdot)), \eta_B(f(\cdot)) \in C(K, L^\phi(\mu)) \). Put
\[
m_A = \phi'((a_1 + a_2)/2), \quad m_B = \phi'((b_1 + b_2)/2).
\]

We have
\[
\phi(a_1 + t) = \phi(a_1) + m_A t \quad \text{for} \ t \in (0, a_2 - a_1)
\]
and
\[
\phi(b_1 + t) = \phi(b_1) + m_B t \quad \text{for} \ t \in (0, b_2 - b_1).
\]

We define functions \( p_A, p_B: L^\phi(\mu) \to \mathbb{R} \) by
\[
p_A(x) = m_A \int \eta_A(x) \, d\mu, \quad p_B(x) = m_B \int \eta_B(x) \, d\mu, \quad x \in L^\phi(\mu).
\]

The functions \( p_A, p_B \) are continuous. Indeed, first consider \( p_A \). Let \( \|x_n - x_0\|_\phi \to 0 \), where \( x_n \in L^\phi(\mu) \). Thus \( \|\eta_A(x_n) - \eta_A(x_0)\|_\phi \to 0 \). Since \( \mu(A) < \infty \), there exists a constant \( c > 0 \) such that \( \|1_A y\|_{L^\phi(\mu)} \leq c\|1_A y\|_\phi \) for all \( y \in L^\phi(\mu) \). Therefore, we obtain \( \|\eta_A(x_n) - \eta_A(x_0)\|_{L^\phi(\mu)} \to 0 \). Hence \( \eta_A(x_n) \, d\mu \to \eta_A(x_0) \, d\mu \), i.e. \( p_A(x_n) \to p_A(x_0) \). Analogously we obtain that \( p_B \) is continuous. Thus \( p_A(f(\cdot)), p_B(f(\cdot)) \in C(K, \mathbb{R}) \).

Now we define \( f_i \in C(K, L^\phi(\mu)), i = 1, 2, \) by
\[
f_i = f + (-1)^i [\alpha_A \eta_A(f) - \alpha_B \eta_B(f)]
\]
where
\[
\alpha_A = \frac{p_B(f)}{1 + |p_A(f)| + |p_B(f)|}, \quad \alpha_B = \frac{p_A(f)}{1 + |p_A(f)| + |p_B(f)|}.
\]

Obviously, \( \alpha_A, \alpha_B \in C(K, \mathbb{R}) \), \( |\alpha_A| \leq 1, |\alpha_B| \leq 1 \).
We have \( f = (f_1 + f_2)/2 \) and \( f_i(k_0) \neq f(k_0) \). We need to check that \( \|f\| \leq 1 \), \( i = 1, 2 \). We have

\[
\int \phi(f_1(k)) \, d\mu = \int_{\Omega \setminus (A \cup B)} \phi(f(k)) \, d\mu + \int_A \phi(f(k)) - m_A \alpha_A \eta_A(f) \, d\mu \\
+ \int_B \phi(f(k)) + m_B \alpha_B \eta_B(f) \, d\mu \\
= \int_{\Omega \setminus (A \cup B)} \phi(f(k)) \, d\mu + \int_{A \cup B} \phi(f(k)) \, d\mu = \int \phi(f(k)) \, d\mu = 1.
\]

Thus \( \|f_1(k)\| = 1 \). Analogously \( \|f_2(k)\| = 1 \) for all \( k \in K \). Therefore, \( f \in \text{ext} \, C(K, L^\Phi(\mu)) \).

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