A NEW PROOF OF A WEIGHTED INEQUALITY FOR THE ERGODIC MAXIMAL FUNCTION

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Abstract. E. Atencia and A. de la Torre proved that the ergodic maximal function operator is bounded on $L^p(\omega)$ if $\omega$ satisfies an appropriate analogue of Muckenhoupt's $A_p$ condition. An alternate proof of this result is given.

Let $(\Omega, \Sigma, \mu)$ be a probability space and suppose $T: \Omega \to \Omega$ is an ergodic, invertible measure preserving transformation. If $\sigma \in L^1(d\mu)$ with $\sigma > 0$ a.e., the two-sided ergodic maximal operator with respect to $\sigma$, denoted $M_\sigma$, is defined for nonnegative integrable $f$ by $(M_\sigma f)(x) = \sup_{m, n \geq 0} A_\sigma(f; m, n, x)$ where

$$A_\sigma(f; m, n, x) = \frac{\sum_{k=-m}^{n} f(T^k x) \sigma(T^k x)}{\sum_{k=m}^{n} \sigma(T^k x)}, \quad x \in \Omega.$$ 

The one-sided maximal operators $M^+_\sigma$ and $M^-_\sigma$ are defined similarly except that the supremum is taken over $m = 0, n > 0$ for $M^+_\sigma$ and $m \geq 0, n = 0$ for $M^-_\sigma$. If $\sigma$ is the constant function equal to one, these operators will be denoted simply as $M, M^+$ and $M^-$, respectively. Note that

$$(1) \quad [(M^+_\sigma + M^-_\sigma)f]/2 \leq M_\sigma f \leq (M^+_\sigma + M^-_\sigma)f.$$ 

E. Atencia and A. de la Torre [1] proved that $M$ is bounded on $L^p(\omega d\mu)$, $1 < p < \infty$, if $\omega \in L^1(d\mu)$ with $\omega > 0$ a.e. and

$$(A_p) \quad \left[ \frac{1}{i} \sum_{k=0}^{i-1} \omega(T^k x) \right]^{1/(p-1)} \leq C \quad \text{a.e.}$$ 

for some constant $C$ and all positive integers $i$. In this note an alternate proof of this result is given. Our proof, an adaptation of that given by M. Christ and R. Fefferman [2] for the Hardy-Littlewood maximal function operator, uses only elementary consequences of the $(A_p)$ condition; in particular, use of the "reverse Hölder" inequality property is avoided.

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The Maximal Ergodic Theorem asserts that $M^+$ is of weak type $(1, 1)$ with respect to $d\mu$. The elegant proof of this result given recently by R. Jones [3] is easily generalized to show that $M^+_\sigma$ is of weak type $(1, 1)$ with respect to the measure $\sigma d\mu$; indeed,

\[(2) \quad \int_{\{x: (M^+ f)(x) > \lambda\}} \sigma(x) \, d\mu(x) \leq \frac{1}{\lambda} \int_\Omega f(x) \sigma(x) \, d\mu(x).\]

Since $T^{-1}$ is also ergodic and measure preserving, it follows that $M^\sigma_\sigma$ is also of weak type $(1, 1)$ with respect to $\sigma d\mu$. Since these operators are clearly of strong type $(\infty, \infty)$, the Marcinkiewicz interpolation theorem shows that these operators are bounded on $L^p(\sigma d\mu), 1 < p < \infty$; from (1) it follows that the same is true of $M_\sigma$.

Set $\sigma(x) = \omega(x)^{-1/\lambda}$. Observe first that if $\omega$ satisfies (A^{\lambda}) then $\sigma \in L^1(d\mu)$. To see this, let $\sigma_n(x) = \min[\sigma(x), n]$ so that $\sigma_n \in L^1(d\mu)$ and (A_p) shows that

\[
\left[ \frac{1}{i} \sum_{k=0}^{i-1} \omega(T^k x) \sigma_n(T^k x) \right]^{p-1} \leq C \quad \text{a.e.}
\]

The Dominated Ergodic and Monotone Convergence Theorems show, upon letting $i \to \infty$, then $n \to \infty$, that

\[(3) \quad \left[ \int_\Omega \omega \, d\mu \right]^{p-1} \left[ \int_\Omega \sigma \, d\mu \right] \leq C.
\]

The main step in the proof is the estimate

\[(4) \quad \int_\Omega (M^+ f)^p \omega \, d\mu \leq B \int_\Omega f^p \omega \, d\mu
\]

for if this is proved, then upon replacing $T$ by $T^{-1}$ we obtain a similar estimate for $M^-$, and (1) then yields the required estimate for $M$.

Without loss of generality assume that $\int_\Omega f \, d\mu = 1$ and then set $E^k = \{ x \in \Omega: (M^+ f)(x) > 4^k \}$ for $k = 1, 2, \ldots$. Now (2) shows that $\mu(E^k) < 1$ and since $T$ is ergodic it follows that for almost all $x \in E^k$ there are positive integers $r = r(x)$ and $s = s(x)$ such that $T^j x \in E^k$ if $-r + 1 \leq j \leq s - 1$ but $T^j x \notin E^k$ for $j = -r$ and $j = s$. Thus, if $B^k_i = \{ x \in E^k: r(x) = 1 \text{ and } s(x) = i \}$, then the sets $T^i(B^k_i)$, $0 \leq j \leq i - 1$, $i = 1, 2, \ldots$, are pairwise disjoint and their union (up to a set of measure zero) is $E^k$.

I wish to thank the referee for pointing out that this decomposition of $E^k$ is the same as that which results from the Kakutani decomposition of $\Omega \setminus E^k$ and in that context the inequalities (5) below were obtained by R. Jones [4].

We need the following lemma but postpone its proof until the end of this paper:

**Lemma.** If $\chi_k$ denotes the characteristic function of $\Omega \setminus E^k$, then for $x \in B^k_i$

\[(5) \quad 4^k < \frac{1}{i} \sum_{j=0}^{i-1} f(T^j x) \leq 2 \cdot 4^k
\]
and

\[ \sum_{j=0}^{i-1} \sigma(T^j x) \leq (2^p C)^{1/(p-1)} \sum_{j=0}^{i-1} \sigma(x_{k+1})(T^j x). \]

Now to obtain (4), write

\[ \int_{\Omega} (M^+ f)^p \omega \, d\mu \leq 4^p \int_{\Omega} \omega \, d\mu + \sum_{k=1}^{\infty} 4^{(k+1)p} \int_{E^k \setminus E^{k+1}} \omega \, d\mu \]

and observe that the first term on the right has the required bound, in view of (3), since by Hölder's inequality

\[ \int_{\Omega} \omega \, d\mu = \left[ \int_{\Omega} \omega \, d\mu \right]^{1/p} \left[ \int_{\Omega} \omega \, d\mu \right]^{-1/p} \left[ \int_{\Omega} f^{p} \omega \, d\mu \right]. \]

On the other hand,

\[ 4^{kp} \int_{E^k \setminus E^{k+1}} \omega \, d\mu \leq 4^{kp} \int_{E^k} \omega \, d\mu = \sum_{i=1}^{\infty} \int_{B_i^k} 4^{kp} \sum_{j=0}^{i-1} \omega(T^j x) \, d\mu \]

and (5) shows that this is bounded by

\[ \sum_{i=1}^{\infty} \int_{B_i^k} \left[ \frac{1}{i} \sum_{j=0}^{i-1} f(T^j x) \right]^p \left[ \sum_{j=0}^{i-1} \omega(T^j x) \right] \, d\mu \]

\[ = \sum_{i=1}^{\infty} \int_{B_i^k} \left[ A_\sigma(f \sigma^{-1}; 0, i - 1) \right]^p \left[ \sum_{j=0}^{i-1} \omega(T^j x) \right] \left[ \frac{1}{i} \sum_{j=0}^{i-1} \sigma(T^j x) \right]^p \, d\mu. \]

Now (A_p) and (6) show that this is bounded by

\[ (2C)^p \sum_{i=1}^{\infty} \int_{B_i^k} \left[ A_\sigma(f \sigma^{-1}; 0, i - 1) \right]^p \left[ \sum_{j=0}^{i-1} \sigma(x_{k+1})(T^j x) \right] \, d\mu \]

\[ \leq (2C)^p \sum_{i=1}^{\infty} \int_{B_i^k} \sum_{j=0}^{i-1} [(M_\sigma f \sigma^{-1})] \left( \sum_{j=0}^{i-1} (\sigma(x_{k+1}))(T^j x) \right) \, d\mu \]

\[ = (2C)^p \int_{E^k \setminus E^{k+1}} \left[ M_\sigma f \sigma^{-1} \right]^p \sigma \, d\mu. \]

Summing over \( k \) and using the boundedness of \( M_\sigma \) on \( L^p(\sigma d\mu) \) shows that the second term in (7) also has the required bound. Thus we have (4).

It remains only to prove the lemma.

The right-hand inequality of (5) is clear since \( T^{-1} x \notin E^k \) implies

\[ \frac{1}{i+1} \sum_{j=-1}^{i-1} f(T^j x) \leq 4^k. \]

Since \( x \in E^k \) there is a positive integer \( n \) such that

\[ \sum_{j=0}^{n-1} f(T^j x) > 4^k n. \]
We may assume that \( n \geq i \), for otherwise, to derive a contradiction, let \( n \) be the largest integer satisfying (8). Then, since \( n < i \), \( T^n x \in E^k \) so there is a positive integer \( m \) such that \( \sum_{n}^{n+m-1} f(T^j x) > 4^k m \). Adding this to (8) then contradicts the maximality of \( n \). On the other hand, if \( n > i \) in (8), then \( T^i x \notin E^k \) implies
\[
\sum_{j=i}^{n-1} f(T^j x) \leq 4^k (n-i).
\]
Subtracting this from (8) shows that (8) also holds for \( n = i \). This proves (5).

Next we show that
\[
(9) \quad \sum_{j=0}^{i-1} \chi_{k+1}(T^j x) > i/2, \quad x \in B_i^k.
\]
Write \( J = \{ j : 0 \leq j \leq i - 1, T^j x \in E^{k+1} \} \) as a disjoint union of maximal blocks of consecutive integers \( J_q, q = 1, \ldots, m \). If \( |J| \) denotes the cardinal of \( J \), then by (5) we have
\[
|J| = \sum_{q-1}^{m} |J_q| < \sum_{q=1}^{m} \left[ 4^{-k-1} \sum_{n \in J_q} f(T^n x) \right] \leq 4^{-k-1} \sum_{j=0}^{i-1} f(T^j x)
\]
and again by (5), this does not exceed \( i/2 \). This proves (9).

The \( (A_p) \) condition and (9) show that
\[
\left[ \sum_{j=0}^{i-1} \omega(T^j x) \right]^{p-1} \left[ \sum_{j=0}^{i-1} \sigma(T^j x) \right] \leq C_i^p \leq 2^{pC} \left[ \sum_{j=0}^{i-1} \chi_{k+1}(T^j x) \right]^{p}
\]
and then Hölder's inequality shows that this does not exceed
\[
2^{pC} \left[ \sum_{j=0}^{i-1} (\omega \chi_{k+1})(T^j x) \right]^{p-1} \left[ \sum_{j=0}^{i-1} (\sigma \chi_{k+1})(T^j x) \right].
\]
From this (6) follows and the lemma is proved.

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