PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 4, December 1986

SOME NEW EXAMPLES OF NONORIENTABLE MINIMAL SURFACES

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ABSTRACT. The classical Henneberg's minimal surface (1875, [**3**, **4**, **11**]) was the unique nonorientable example known until 1981, when Meeks [**6**] exhibited the first example of a nonorientable, regular, complete, minimal surface of finite total curvature -6π.

In this paper, we study the nonorientable, regular, complete minimal surfaces of finite total curvature and give some examples of punctured projective planes regularly and minimally immersed in $\mathbf{R}^3$ and $\mathbf{R}^n$.

1. Nonorientable minimal surfaces in $\mathbf{R}^n$. We consider surfaces S in $\mathbf{R}^n$ defined by maps $X: M \to \mathbf{R}^n$, where M is a two-dimensional manifold.

For the study of a nonorientable, connected surface S we take the double surface $\tilde{S}$ given by $\tilde{X}: \tilde{M} \to \mathbf{R}^n$, where $\Pi: \tilde{M} \to M$ is the oriented two-sheeted covering of M, and $\tilde{X} = X \circ \Pi$. We have an involution $\tilde{I}: \tilde{M} \to \tilde{M}$ without fixed points and a conformal structure on $\tilde{M}$ such that $\tilde{I}$ is antiholomorphic.

We have a representation theorem for the nonorientable minimal surfaces:

THEOREM 1.1. *Let S be a nonorientable regular connected minimal surface in $\mathbf{R}^n$. The double surface $\tilde{S}$ is a minimal surface in $\mathbf{R}^n$ defined by $\tilde{X}: \tilde{M} \to \mathbf{R}^n$,*

$$\tilde{X}(p) = \operatorname{Re} \int_{p_0}^{p} \tilde{\phi}(\varsigma)\, d\varsigma, \qquad p_0, p \in \tilde{M}, \tag{1.1}$$

with $\alpha = \tilde{\phi}(\varsigma) d\varsigma$ such that $\tilde{I}^* \alpha = \overline{\alpha}$ (that is, $\tilde{I}^* \alpha_k = \overline{\alpha}_k$, $1 \le k \le n$).

Conversely, if $\tilde{S}$ is a regular orientable connected minimal surface in $\mathbf{R}^n$ given by (1.1) and if there is an antiholomorphic involution $\tilde{I}: \tilde{M} \to \tilde{M}$ without fixed points such that $\tilde{I}^* \alpha = \overline{\alpha}$, then $\tilde{S}$ is the double surface of a regular nonorientable minimal surface in $\mathbf{R}^n$.

The consequences of the condition $\tilde{I}^* \alpha = \overline{\alpha}$ with respect to the various forms of representation for minimal surfaces in $\mathbf{R}^n$ are

COROLLARY 1.2 (MEEKS [**6**]). *Let f and g be the functions of Weierstrass's representation of an orientable, regular, connected minimal surface $\tilde{S}$ in $\mathbf{R}^3$, that is*

$$\tilde{X}(p) = \operatorname{Re} \int_{p_0}^{p} \frac{f(\varsigma)}{2} (1 - g^2(\varsigma), i(1 + g^2(\varsigma)), 2g(\varsigma))\, d\varsigma, \qquad p_0, p \in \tilde{M}.$$

The surface $\tilde{S}$ is the double surface of a nonorientable, regular minimal surface in $\mathbf{R}^3$ if and only if

$$\begin{cases} \text{(i)} & g(\tilde{I}(p)) = -1/\overline{g(p)}, \qquad p \in \tilde{M}, \text{ and} \\ \text{(ii)} & \tilde{I}^*(\omega) = -g^2 \omega, \qquad \omega = f(z)\, dz, \end{cases} \tag{1.2}$$

for some antiholomorphic involution $\tilde{I}: \tilde{M} \to \tilde{M}$ without fixed points.

Received by the editors October 31, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53A10; Secondary 30A68.

COROLLARY 1.3. *Let $\tilde{S}$ be an orientable, regular, connected minimal surface in $\mathbf{R}^4$ given by $\tilde{X}\colon \tilde{M} \to \mathbf{R}^4$,*

$$\tilde{X}(p) = \operatorname{Re} \int_{p_0}^{p} (1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2))\omega, \qquad p_0, p \in \tilde{M}.$$

Then, $\tilde{S}$ is the double surface of a nonorientable minimal surface in $\mathbf{R}^4$ if and only if, for some antiholomorphic involution $\tilde{I}$ on $\tilde{M}$ without fixed points,

$$\begin{cases} \text{(i)} & g_k(\tilde{I}(p)) = -1/\overline{g_k(p)}, \qquad k = 1, 2, \text{ and} \\ \text{(ii)} & \tilde{I}^*\omega = \overline{g_1 g_2 \omega}, \qquad \omega = f(z)\, dz, \end{cases} \tag{1.3}$$

If $C(\tilde{S})$ is the total curvature of $\tilde{S}$, we can define the total curvature of S by

$$C(S) = C(\tilde{S})/2. \tag{1.4}$$

In what follows, the double surface $\tilde{S}$ will always be connected, complete and of finite total curvature $C(\tilde{S}) = 2\pi\tilde{m}$. From Chern and Osserman's theorem [**1**] we have $\tilde{M}$ conformally equivalent to a compact Riemann surface of genus $\tilde{\gamma}$ punctured at $\tilde{r} = 2r$ points $\{p_1, p_2, \ldots, p_r, q_1, \ldots, q_r\}$, with $\tilde{I}(p_j) = q_j$, $\tilde{I}(p_j)$ being the extension of $\tilde{I}$ to p_j, $1 \le j \le r$. The function $\tilde{\phi}$ of (1.1) has poles of order m_j at p_j and q_j, and

$$\tilde{m} - 2\sum_{j=1}^{r} m_j = 2\tilde{\gamma} - 2. \tag{1.5}$$

The Chern and Osserman inequality for nonorientable minimal surfaces is

$$C(S) \le 2\pi(\chi(M) - r), \tag{1.6}$$

with $\chi(M)$ the Euler characteristic of M.

A nonorientable version of Gackstatter's theorem [**2**] gives an estimate for the dimension of a nonorientable, complete, minimal surface S with total curvature $C(S) = -2\pi m$, r ends and genus γ:

$$\operatorname{Dim}(S) \le 2m - 2\gamma - r + 3. \tag{1.7}$$

We have

PROPOSITION 1.4. *The total curvature of a nonorientable, regular, complete minimal surface is at most -4π.*

PROOF. From (1.7) and the fact that $\operatorname{Dim} S \ge 3$, $2m \ge 4$.

2. Genus one nonorientable complete minimal surfaces in $\mathbf{R}^n$. A characterization of the complete minimal surfaces of finite total curvature and genus

one can be set up combining Theorem 1.1 and Hoffman and Osserman's theorem [5]:

THEOREM 2.1. *Let $\tilde{M}$ be the complex plane minus $2r-1$ points $\{0, z_1, \dots, z_{r-1}, -1/\overline{z}_1, \dots, -1/\overline{z}_{r-1}\}$. Let the complex vector $\phi(\varsigma)$ be of the form*

$$\phi(\varsigma) = F(\varsigma)(p_1(\varsigma), \dots, p_n(\varsigma)),$$

with

$$F(\varsigma) = \left(\varsigma^{\nu_0} \prod_{k=1}^{r-1} (\varsigma - z_k)^{\nu_k} (\varsigma + \frac{1}{\overline{z}_k})^{\nu_k}\right)^{-1},$$

$p_j(\varsigma)$ satisfying

(i) *$p_j(\varsigma)$ is a polynomial,*

(ii) *the maximum degree of p_j is $2m$,*

(iii) *the p_j's have no common factor,*

(iv) $\sum_{j=1}^{n} p_j^2 \equiv 0$,

(v) $(-1)^{m+1}\overline{\varsigma}^{2m} p_j(-1/\overline{\varsigma}) = \overline{p_j(\varsigma)}$, $1 \le j \le m$.

Suppose that the ν_k's satisfy $\sum_{k=0}^{r-1} \nu_k = m+1$, $\nu_k \ge 2$, $0 \le k \le r-1$ and $\operatorname{Re} \int_\gamma \phi = 0$, for any simple closed curve γ in $\tilde{M}$. Then $\tilde{X}(p) = \operatorname{Re} \int_{p_0}^{p} \phi(\varsigma)\, d\varsigma$ defines a double minimal surface associated to a nonorientable regular complete minimal surface in $\mathbf{R}^n$ of genus one, r ends and total curvature $-2\pi m$.

Conversely, every genus one nonorientable minimal surface in $\mathbf{R}^n$ admits a representation as mentioned above.

PROOF. The double surface associated to a genus one nonorientable minimal surface is conformally equivalent to the complex plane $\hat{\mathbf{C}}$ minus $2r$ points with the involution $\tilde{I}: \mathbf{C} \to \mathbf{C}$ given by $\tilde{I}(z) = -1/\overline{z}$, that is,

$$\tilde{M} = \mathbf{C} - \{0, z_1, \dots, z_{r-1}, -1/\overline{z}_1, \dots, -1/\overline{z}_{r-1}\}.$$

From Hoffman and Osserman's result, $\phi = F \cdot (p_1, \dots, p_n)$, p_j polynomials satisfying (i)–(iv).

The map $\tilde{X}: \tilde{M} \to \mathbf{R}^n$ defines a double surface if and only if $\tilde{I}^*(\phi_j(\varsigma)\, d\varsigma) = \overline{\phi_j(\varsigma)\, d\varsigma}$, $1 \le j \le n$; from this fact and (1.5), we have $\sum_{k=0}^{r-1} \nu_k = m+1$ and (v). □

We study initially surfaces whose total curvature is the upper bound given by Proposition 1.4.

THEOREM 2.2. *Let S be a nonorientable, regular, complete minimal surface of finite total curvature -4π. Then S is a minimal immersion of the projective plane minus one point and lies fully in $\mathbf{R}^4$; any two such minimal surfaces are similar.*

PROOF. From (1.5), and (1.6) and (1.7), using $\chi(M) = 2 - \gamma - r$, we have $\gamma = 1$, $r = 1$ and $m_1 = 3$. The double surface has genus zero and two ends; thus, by Theorem 2.1, it can be given by $\tilde{X}: C - \{0\} \to \mathbf{R}^n$,

$$\tilde{X}_j(p) = \operatorname{Re} \int_{p_0}^{p} \phi_j(\varsigma)\, d\varsigma, \qquad p_0, p \in \tilde{M} = \mathbf{C} - \{0\},\ \phi_j(\varsigma) = p_j(\varsigma)/\varsigma^3,$$

p_j satisfying conditions (i)–(v).

Setting $p_j(\varsigma) = a_j\varsigma^4 + b_j\varsigma^3 + c_j\varsigma^2 + d_j\varsigma + e_j$, we can verify that $a_j = -e_j$, $b_j = \overline{d}_j$, $c_j = 0$ and that $A_1 = (\operatorname{Re} a_1, \dots, \operatorname{Re} a_n)$, $A_2 = (\operatorname{Im} a_1, \dots, \operatorname{Im} a_n)$, $B_1 = (\operatorname{Re} b_1, \dots, \operatorname{Re} b_n)$ and $B_2 = (\operatorname{Im} b_1, \dots, \operatorname{Im} b_n)$ are real orthogonal vectors of the same length.

Thus, $A = (1, i, 0, 0)$ and $B = (0, 0, 1, i)$ give a solution in $\mathbf{R}^4$; two solutions $\tilde{X}$ and $\tilde{Y}$ are such that $\tilde{Y} = \lambda M\tilde{X} + X_0$, with $\lambda \in \mathbf{R}$ and M a real orthogonal matrix.

Immediately, we have Meeks' theorem:

COROLLARY 2.3. *The total curvature of a nonorientable, regular, complete minimal surface in* $\mathbf{R}^3$ *is at most* -6π.

The total curvature of an orientable minimal surface in $\mathbf{R}^3$ is of the form $C(\tilde{S}) = -4\pi \cdot \operatorname{degree}(g)$, with g the meromorphic function of the Weierstrass representation. Thus, for nonorientable minimal surfaces,

$$C(S) = -2\pi \cdot \operatorname{degree}(g), \tag{2.1}$$

with $g\colon \tilde{M} \to \mathbf{C}$ the function of the Weierstrass representation of the double surface $\tilde{S}$, which is a rational function when it has finite degree and the genus of $\tilde{M}$ is zero.

The next result is very important for the construction of examples:

PROPOSITION 2.4. *Let g be a rational complex function satisfying $g(-1/\overline{z}) = -1/\overline{g(z)}$. Then, the degree of g is odd and g is of the form*

$$g(z) = cz^\alpha \frac{\prod_{j=1}^m (z - a_j)}{\prod_{j=1}^m (z + 1/\overline{a}_j)}, \quad \textit{with } |c| \prod_{j=1}^m |a_j| = 1.$$

PROOF. Let g be a rational function, and let

$$g(z) = cz^\alpha \frac{\prod_{j=1}^m (z - a_j)}{\prod_{k=1}^n (z - b_k)}, \qquad a_j \neq b_k \text{ for any } j, k, a_j \neq 0,\ b_k \neq 0,$$

where the factors appear with multiplicities. From the condition

$$g(-1/\overline{z}) = -1/\overline{g(z)},$$

with some calculations, we complete the proof. □

PROPOSITION 2.5. *The total curvature of a nonorientable regular complete minimal surface in* $\mathbf{R}^3$ *of genus one is of the form $C(S) = -2\pi m$, m odd, $m \geq 3$.*

PROOF. This follows immediately from Proposition 2.4. □

Meeks [**6**] has proved that there exists a unique nonorientable regular complete minimal surface in $\mathbf{R}^3$ of total curvature -6π. We extend Meeks' surface in the sense of

THEOREM 2.6. *There exists a nonorientable regular complete minimal surface in* $\mathbf{R}^3$ *of genus one, one end and total curvature $-2\pi m$, for any m odd, $m \geq 3$.*

PROOF. With straightforward calculations, one can show that

$$f(z) = i(z+1)^2/z^{m+1} \quad \text{and} \quad g(z) = z^{m-1}(z-1)/(z+1)$$

are the functions of the Weierstrass representation of a double minimal surface $\tilde{S}$; that is, the map

$$\tilde{X}(p) = \operatorname{Re} \int_{p_0}^{p} \frac{f}{2}(1 - g^2, i(1 + g^2), 2g)\, d\varsigma, \qquad p_0, p \in \mathbf{C} - \{0\},$$

satisfies (i)–(v) of Theorem 2.1. The surfaces obtained are regular and complete. □

REMARKS. 1. If $n = 3$ the surface of Theorem 2.6 is exactly Meeks' surface.

2. The surface is an infinite Moebius band with $(m-1)/2$ twists and the image of $|z| = 1$ is a circle centered at $(0, -2/(m-1), 0)$ with radius $2/(m-1)$ covered $(m-1)$ times.

3. The normals to the surface along $|z| = 1$ have the third component $N_3|_{|z|=1} = \operatorname{Re}(z)|_{|z|=1}$.

The Chern-Osserman inequality for nonorientable minimal surfaces of total curvature -6π gives us the relation $\gamma + 2r \leq 5$; hence, if $\gamma = 1$, we may have $r = 1$ or $r = 2$. Meeks has proved that the case $r = 2$, $C = -6\pi$ cannot occur. Thus, we have

PROPOSITION 2.7. *The total curvature of a nonorientable regular complete minimal surface in* $\mathbf{R}^3$ *of genus one and two ends is at most* -10π.

For $C = -10\pi$, we have

THEOREM 2.8. *There exists a nonorientable regular complete minimal surface in* $\mathbf{R}^3$ *of genus one, two ends and total curvature* -10π.

PROOF. The double surface associated to such a surface will be an orientable surface of genus zero, four ends with total curvature -20π and, by Theorem 2.1, it may be given by

$$\tilde{X}\colon \mathbf{C} - \{0, 1, -1\} \to \mathbf{R}^3, \quad \tilde{I}\colon \mathbf{C} \to \mathbf{C}, \quad \tilde{I}(z) = -1/\bar{z}.$$

Let $f(z) = a(\bar{b}^2 z^2 - 1)^2/z^2(z-1)^4(z+1)^4$ and $g(z) = \alpha\bar{b}^2 z^3(z^2 - b^2)/(\bar{b}^2 z^2 - 1)$; f and g are the functions of the Weierstrass representation of a double minimal surface if and only if

$$|\alpha|\,|b|^2 = 1 \quad \text{and} \quad a = -\bar{a}\bar{\alpha}b^4 \quad \text{or} \quad a\alpha\bar{b}^2 = -\bar{a}\bar{\alpha}b^2. \tag{2.2}$$

The completeness and regularity can easily be verified.

To choose α, a and b such that

$$\operatorname{Re}\int_\gamma \frac{f(x)}{2}(1 - g^2(z), i(1 + g^2(z)), 2g(z))\,dz = \overrightarrow{0}$$

for any closed curve γ in $\mathbf{C} - \{0, 1, -1\}$, we observe that, by Cauchy's integral formula, for simple closed curves γ_0 and γ_1 around $z = 0$ and $z = 1$, respectively,

$$\int_{\gamma_0} \frac{p_j(\varsigma)\,d\varsigma}{\varsigma^2(\varsigma-1)^4(\varsigma+1)^4} = 2\pi i p_j'(0),$$

and

$$\int_{\gamma_1} \frac{p_j(\varsigma)\,d\varsigma}{\varsigma^2(\varsigma-1)^4(\varsigma+1)^4} = \frac{2\pi i}{3!}\cdot\frac{1}{16}[p_j'''(1) - 12p_j''(1) + 57p_j'(1) - 105p_j(1)].$$

Calling $R_j = p_j'''(1) - 12p_j''(1) + 57p_j' - 105p_j(1)$, $j = 1, 2, 3$, we have

$$\begin{aligned} R_1 &= -105(a - \bar{a}) + 30(a\bar{b}^2 - \bar{a}b^2) + 3(a\bar{b}^4 - \bar{a}b^4), \\ R_2 &= i[-105(a + \bar{a}) + 30(a\bar{b}^2 + \bar{a}b^2) + 3(a\bar{b}^4 + \bar{a}b^4)], \\ R_3 &= 0. \end{aligned}$$

Thus, $\mathrm{Re}[2\pi i R_j] = 0$, $j = 1, 2$ if $-105a + 30a\bar{b}^2 + 3a\bar{b}^4 = 0$.

For each b, $\alpha = -1/\bar{b}^2$, $a = i$ satisfy (2.2) and

$$\tilde{X}(p) = \mathrm{Re} \int_{p_0}^{p} \frac{f}{2}(1 - g^2, i(1 + g^2), 2g)\, d\varsigma$$

gives the double surface. □

We investigate the problem of the regular minimal surfaces in $\mathbf{R}^3$ of finite total curvature, genus one and three ends, and, we have

THEOREM 2.9. *There exists a nonorientable regular complete minimal surface in $\mathbf{R}^3$ of genus one, three ends and total curvature -14π.*

PROOF. We construct the double surface $\tilde{S}$ defined by $\tilde{X}\colon \mathbf{C} - \{0, 1, -1, i, -i\} \to \mathbf{R}^3$ with the functions f and g of the Weierstrass representation (1.2) given by $g(z) = a\bar{b}^4 z^3(z^4 - b^4)/(\bar{b}^4 z^4 - 1)$ and $f(z) = \alpha(\bar{b}^4 z^4 - 1)^2/z^2(z^2 - 1)^3(z^2 + 1)^3$. The functions f and g satisfy (1.2) if and only if

$$|a|\,|b|^4 = 1, \qquad \alpha = \bar{\alpha}\bar{a}^2 b^8. \tag{2.3}$$

With some calculations, we can verify that the real parts of the integrals

$$\int_\gamma \frac{p_j(z)\, dz}{z^2(z^4 - 1)^3},$$

where γ is a simple closed curve, are null for any j, $j = 1, 2, 3$, provided b satisfies the following condition: $3b^8 + 10b^4 - 45 = 0$. Thus, $b^4 \in \mathbf{R}$.

Choosing $a = 1/b^4$ and $\alpha = 1$ the conditions (2.3) are verified. The completeness and regularity can be easily checked. □

In $\mathbf{R}^4$, exploring the properties of the functions g_1, g_2 of the representation (1.3), we have

THEOREM 2.10. *For any integer m, $m \geq 2$, there exists a nonorientable regular complete minimal surface S in $\mathbf{R}^4$ of genus one, one end and total curvature $-2\pi m$.*

PROOF. The double surface $\tilde{S}$ associated to S has total curvature $-4\pi m = -2\pi\tilde{m}$, $\tilde{m} = 2m$, genus zero and two ends. We set the functions f, g_1, g_2 in the representation (1.3) by $f(z) = z^{m-1}$, $g_1(z) = \varepsilon_1/z^{k_1}$, $g_2(z) = \varepsilon_2/z^{k_2}$, with $\tilde{m} = 2m = k_1 + k_2$, k_1, k_2 odd, $k_1, k_2 \geq 1$, and $\tilde{M} = \mathbf{C} - \{0\}$.

The functions f, g_1 and g_2 satisfy (i) and (ii) of Corollary 1.3, if and only if

$$|\varepsilon_1| = |\varepsilon_2| = 1, \qquad \varepsilon_1\varepsilon_2 = (-1)^{m-1}. \tag{2.4}$$

To verify that $\mathrm{Re} \int_\gamma \phi(\varsigma)\, d\varsigma = \overrightarrow{0}$, for any closed curve in $\mathbf{C} - \{0\}$, we observe that

$$\begin{aligned}
\phi_1(\varsigma) &= f(1 + g_1 g_2) = z^{m-1} + \varepsilon_1\varepsilon_2/z^{m+1},\\
\phi_2(\varsigma) &= if(1 - g_1 g_2) = i(z^{m-1} - \varepsilon_1\varepsilon_2/z^{m+1}),\\
\phi_3(\varsigma) &= f(g_1 - g_2) = (\varepsilon_1 z^{k_2} - \varepsilon_2 z^{k_1})/z^{m+1},\\
\phi_4(\varsigma) &= -if(g_1 + g_2) = -i(\varepsilon_1 z^{k_2} + \varepsilon_2 z^{k_1})/z^{m+1},
\end{aligned}$$

and rewriting $k_1 = m - n$, $k_2 = m + n$, $n \geq 1$, we have $\int_\gamma \phi_j = 0$, $j = 1, \ldots, 4$. Thus, the double surface $\tilde{S}$ is defined by

$$\tilde{X}(p) = \operatorname{Re} \int_{p_0}^{p} \phi(\varsigma)\, d\varsigma, \qquad p_0, p \in \mathbf{C} - \{0\}. \quad \square$$

For $\gamma = 1$, $r = 1$, the dimension of S satisfies $\operatorname{Dim}(S) \leq 2m$. We will show that this upper bound is sharp.

THEOREM 2.11. *For any integer m, $m \geq 2$, there exists a nonorientable regular complete minimal surface S in $\mathbf{R}^{2m}$ of genus one, one end and total curvature $-2\pi m$ that lies fully in $\mathbf{R}^{2m}$.*

PROOF. The double surface $\tilde{S}$ associated to S, by Theorem 2.1, is given by $\tilde{X}\colon \mathbf{C} - \{0\} \to \mathbf{R}^n$, $\tilde{X}(p) = \operatorname{Re} \int_{p_0}^{p} \phi(\varsigma)\, d\varsigma$, with $\phi_j(\varsigma) = p_j(\varsigma)/\varsigma^{m+1}$, $p_j(\varsigma)$ satisfying (i)–(v).

Particularly, condition (v) is equivalent to

$$(-1)^{m+1}\bar{\varsigma}^{2m} p_j(-1/\bar{\varsigma}) = \overline{p_j(\varsigma)}, \qquad 1 \leq j \leq n. \tag{2.5}$$

Taking $p_j(\varsigma) = a_0^j + a_1^j \varsigma + \cdots + a_{2m-1}^j \varsigma^{2m-1} + a_{2m}^j \varsigma^{2m}$ in (2.5) we obtain

$$(-1)^{m+1}(a_0^j \bar{\varsigma}^{2m} - a_1^j \bar{\varsigma}^{2m-1} + \cdots + a_{2m}^j) = (\bar{a}_{2m}^j \bar{\varsigma}^{2m} + \bar{a}_{2m-1}^j \bar{\varsigma}^{2m-1} + \cdots + \bar{a}_0^j).$$

Then, there are two possibilities:

(a) $(m+1)$ even: $a_0^j = \bar{a}_{2m}^j$, $-a_1^j = \bar{a}_{2m-1}^j$, $\ldots$, $a_{m-1}^j = \bar{a}_{m+1}^j$, $-a_m^j = \bar{a}_m^j$,

(b) $(m+1)$ odd: $a_0^j = -\bar{a}_{2m}^j$, $a_1^j = \bar{a}_{2m-1}^j$, $\ldots$, $a_{m-1}^j = \bar{a}_{m+1}^j$, $a_m^j = -\bar{a}_m^j$.

The integrals $\int_\gamma \phi_j(\varsigma)\, d\varsigma$, γ a closed curve in $\mathbf{C} - \{0\}$, should have null real part; therefore, $\operatorname{Re}\{2\pi i \cdot a_m^j\} = 0$, $1 \leq j \leq n$, that is $\operatorname{Im} a_m^j = 0$, $1 \leq j \leq n$. Comparing with the last condition in (a) or (b), we have $a_m^j = 0$, $1 \leq j \leq n$.

To determine solutions in $\mathbf{R}^n$, we should find complex vectors $A_k = (a_k^1, \ldots, a_k^n)$, $0 \leq k \leq m-1$, such that the corresponding polynomials satisfy $\sum_{j=1}^n p_j^2(\varsigma) \equiv 0$.

It is easy to verify that $A_0 = (1, i, 0, \ldots, 0)$, $A_1 = (0, 0, 1, i, 0, \ldots, 0), \ldots$, $A_{m-1} = (0, \ldots, 0, 1, i)$ give a solution in $\mathbf{C}^{2m}$ if m is even and $A_0 = (1, i, 0, \ldots, 0)$, $A_2 = (0, 0, 1, i, 0, \ldots, 0), \ldots, A_{m-2} = (0, 0, \ldots, 0, \sqrt{2}, i\sqrt{2}, 0, 0)$, $A_{m-1} = (0, 0, \ldots, 0, 1, i)$ give a solution if m is odd. $\square$

With these results, we have partial answers for the problem of existence of nonorientable, regular, minimal surfaces in $\mathbf{R}^3$ and $\mathbf{R}^n$ posed by Nitsche [**7**, 1965]. We do not know, for instance, an example of a genus two nonorientable minimal surface in $\mathbf{R}^3$, that is, a minimal Klein bottle in $\mathbf{R}^3$.

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