GENERAL POSITION THEOREMS FOR GENERALIZED MANIFOLDS

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ABSTRACT. It is an open question as to whether a generalized \( n \)-manifold, \( n \geq 5 \), that satisfies the disjoint disks property is a topological manifold. In this paper it is shown that any such space \( X \) satisfies general position properties for maps of polyhedra into \( X \).

The program to establish a topological characterization of euclidean space has been stalled, at least temporarily, by the discovery that Quinn's obstruction to resolving a generalized manifold may not be categorically zero as first asserted in [6]. (See [7].) Thus Cannon's conjecture [3] that a generalized \( n \)-manifold \( X \), \( n \geq 5 \), is a topological manifold if and only if \( X \) satisfies the disjoint disks property remains open. Quinn's results [6, 7], however, do produce an obstruction \( \sigma(X) \) that vanishes if and only if \( X \) is a topological manifold, although it is not known at this time whether the obstruction can be nonzero.

In this paper we show (Theorem 3) that any generalized \( n \)-manifold \( X \), \( n \geq 5 \), having the disjoint disks property also satisfies general position properties for maps of arbitrary polyhedra into \( X \). Thus, in a dimension theory sense \( X \) behaves like a topological manifold. Theorems of this type have also been obtained by John Walsh [9].

DEFINITIONS. A generalized \( n \)-manifold, \( n \)-gm, is a euclidean neighborhood retract \( X \) that is also a homology \( n \)-manifold; that is, \( H_*(X,X-x) \cong H_*(\mathbb{R}^n,\mathbb{R}^n-0) \) for all \( x \in X \). (All homology is understood to have integer coefficients.) The space \( X \) is said to satisfy the disjoint \( k \)-disks property, \( DDP \) (or, simply, the disjoint disks property, \( DDP \), when \( k = 2 \)) if for every pair of maps \( f_1, f_2 : D_k \to X \) of the unit \( k \)-disk \( D_k \) of \( \mathbb{R}^k \) into \( X \) and \( \epsilon > 0 \), there are maps \( f'_1, f'_2 : D_k \to X \) such that \( d(f'_1, f'_2) < \epsilon \) and \( f'_1(D_k) \cap f'_2(D_k) = \emptyset \). If \( f, g : A \to X \) and \( \epsilon > 0 \), then \( f \simeq_\epsilon g \) means \( f \) is \( \epsilon \)-homotopic to \( g \). A subset \( A \) of \( X \) is said to be 1-LCC in \( X \) (1-locally coconnected in \( X \)) if for every \( a \in A \) and neighborhood \( U \) of \( a \) in \( X \) there is a neighborhood \( V \) of \( a \) in \( X \) such that the inclusion induced homomorphism \( \pi_1(V-A) \to \pi_1(U-A) \) is zero. A map (or embedding) \( f : A \to X \) is said to be 1-LCC provided \( f(A) \) is 1-LCC in \( X \).

Main results. In [3] Cannon shows that if \( X \) is an \( n \)-gm, \( n \geq 5 \), having the DDP, then an arbitrary map of a 2-disk (hence, a 2-dimensional polyhedron) into...
X can be approximated by 1-LCC embeddings. Our first result is

**Theorem 1.** If $X$ is an $n$-gm, $n \geq 5$, satisfying the DDP, $P$ is a $k$-dimensional polyhedron, $2k + 1 \leq n$, and $f : P \to X$, then $f$ can be approximated by 1-LCC embeddings.

We show that an $n$-gm $X$ with the DDP is never "ghastly" [4].

**Theorem 2.** Suppose $X$ is an $n$-gm, $n \geq 5$, having the DDP, $P$ is a $k$-dimensional polyhedron, and $f : P \to X$ is a map. Then $f$ can be approximated by a map $f' : P \to X$ such that

1. $\dim f'(P) \leq k$, and
2. $f'(P)$ is 1-LCC if $k \leq n - 3$.

Finally, we obtain the main general position property.

**Theorem 3.** Suppose $X$ is an $n$-gm, $n \geq 5$, with the DDP, $P$ and $Q$ are polyhedra of dimensions $p$ and $q$, respectively, and $f : P \to X$ and $g : Q \to X$ are maps. Then $f$ and $g$ can be approximated by maps $f'$ and $g'$ such that

1. $\dim [f'(P) \cap g'(Q)] \leq p + q - n$, and
2. $f'(P) \cap g'(Q)$ is 1-LCC in $X$ if $p + q - n \leq n - 3$.

**Proofs of the main results.** We start with a well-known fact (proved, for example, in [2, 8]).

**Lemma 0.** A locally compact, finite-dimensional, separable metric space $X$ admits a 1-LCC embedding in some euclidean space.

**Lemma 1.** Suppose $X$ is an $n$-dimensional, 1-LCC closed subset of $\mathbb{R}^m$, $m - n \geq 3$. Then there are $F_n$ subsets $X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that

1. $\dim X_i \leq i$, and
2. $\dim (X_i - X_j) \leq i - j - 1$.

**Proof.** Let $L_k \subset \mathbb{R}^m$ denote the Menger space:

$$L_k = \{x \in \mathbb{R}^m | \text{at least } m - k \text{ coordinates of } x \text{ are rational}\}.$$

Then $\dim L_k = k$, $L_k$ is a countable union of $k$-dimensional hyperplanes, and $\dim (\mathbb{R}^m - L_k) = m - k - 1$ [5]. Also $L_0 \subset L_1 \subset \cdots \subset L_m = \mathbb{R}^m$ and $\dim (L_r - L_s) = r - s - 1$. Since $X$ is 1-LCC in $\mathbb{R}^m$, we can assume that $L_k \cap X = \emptyset$ for $k \leq m - n - 1$ [2, 8].

Set $X_i = X \cap L_{m-n+i}$. Then

1. $\dim X_i = \dim (X \cap L_{m-n+i})$
   $= \dim (X \cap (L_{m-n+i} - L_{m-n-1}))$
   $\leq (m - n + i) - (m - n - 1) - 1 = i$,

and

2. $\dim (X_i - X_j) = \dim (X \cap (L_{m-n+i} - L_{m-n+j}))$
   $\leq \dim (L_{m-n+i} - L_{m-n+j}) = i - j - 1$.

Throughout the rest of this paper $X$ is an $n$-gm satisfying the DDP, 1-LCC embedded in $\mathbb{R}^m$ for some $m \geq n + 3$. 
Lemma 2. There are \( F_\sigma \) subsets \( X_0 \subset X_1 \subset \cdots \subset X_n = X \) such that
1. \( \dim X_i \leq i \),
2. \( \dim(X_i - X_j) \leq i - j - 1 \), and
3. \( X_i \) is 1-LCC in \( X \) for \( i \leq n - 3 \).

Proof. Let \( \{ f_i \}_{i=1}^{\infty} \) be a countable dense subset of \( \text{Map}(D^2, X) \), the space of maps (with the compact-open topology), consisting of 1-LCC embeddings \([3]\). In the proof of Lemma 1, require that \( L_i \cap f_i(D^2) = \emptyset \) for \( k \leq m - 3 \) and all \( i \) (using the fact that the 1-LCC property is transitive). Then \( X_i \cap f_j(D^2) = \emptyset \) for \( i \leq n - 3 \). Thus \( X_i \) is 1-LCC in \( X \) for \( i \leq n - 3 \).

Lemma 3. Let \( X_0 \subset X_1 \subset \cdots \subset X_n = X \) be \( F_\sigma \) subsets of \( X \) satisfying the conclusion of Lemma 2. Then \( \pi_k(X, X - X_i) = 0 \) for \( k < n - i \).

Proof. By local duality in \( X \) \([1, 10]\), \( H_k(X, X - X_i) = 0 \) for \( k < n - i \). If \( i = n - 1 \) or \( n - 2 \), then \( \pi_k(X, X - X_i) = H_k(X, X - X_i) = 0 \). If \( i \leq n - 3 \), then \( X_i \) is 1-LCC in \( X \) so that \( \pi_2(X, X - X_i) = 0 \). Thus by the Hurewicz isomorphism theorem, \( \pi_k(X, X - X_i) = 0 \) for \( i \leq n - 3 \) and \( k < n - i \). (We may assume that \( X \), and hence \( X - X_i \), \( i \leq n - 3 \), are simply connected by passing to the universal cover of \( X \).)

Lemma 4. Suppose \( f: D^k \to X \) is a map and \( \varepsilon > 0 \). Then \( f \simeq_{\varepsilon} f' \) such that \( f'(D^k) \cap X_{n-k-1} = \emptyset \) and \( f' \) is 1-LCC, if \( k \leq n - 3 \).

Proof. To get \( f'(D^k) \cap X_{n-k-1} = \emptyset \), apply Lemma 3 locally, using induction on the skeleta of a fine triangulation of \( D^k \). Similarly, if \( D \) is a 1-LCC 2-cell in \( X \), then for a relative neighborhood \( U \) of \( \text{Int} D \), \( \pi_2(U, U - \text{Int} D) = 0 \) and \( H_i(U, U - \text{Int} D) = 0 \) for \( i \leq n - 3 \). Thus for \( r \leq n - 3 \) any map \( g: (D^r, \text{Bd} D^r) \to (U, U - \text{Int} D) \) is homotopic rel \( \text{Bd} D^r \) to a map \( g': D^r \to U - \text{Int} D \). Applying this argument inductively on the skeleta of a fine triangulation of \( D^k \), we can get \( f \simeq_{\varepsilon} f' \) such that \( f'(D^k) \cap D = \emptyset \) for any 1-LCC 2-cell \( D \) in \( X \). Thus we may simultaneously obtain \( f \simeq_{\varepsilon} f' \) so that \( f' \) is 1-LCC and \( f'(D^k) \cap X_{n-k-1} = \emptyset \).

Corollary. If \( f: D^k \to X \) and \( \varepsilon > 0 \), then \( f \simeq_{\varepsilon} f' \) where \( f' \) is 1-LCC, if \( k \leq n - 3 \), and \( \dim f'(D^k) \leq k \).

The proof of Theorem 2 follows immediately.

Lemma 5. \( X \) satisfies \( DD_kP \) for \( k < n/2 \).

Proof. Given \( f, g: D^k \to X \), apply the Corollary to get \( f \simeq_{\varepsilon} f' \) where \( f' \) is 1-LCC and \( \dim f'(D^k) \leq k \). Then for any open set \( U \) of \( X \), \( \pi_i(U, U - f'(D^k)) = 0 \) for \( i \leq n - k - 1 \). If \( k < n/2 \), then \( k \leq n - k - 1 \) and, hence, by a now familiar argument, for any \( \varepsilon > 0 \), \( g \simeq_{\varepsilon} g' \) where \( g'(D^k) \cap \pi'(D^k) = \emptyset \).

The proof of Theorem 1 now follows easily as in \([3]\).

Proof of Theorem 3. It is clearly sufficient to prove the theorem when \( P \) is a \( p \)-simplex and \( Q \) is a \( q \)-simplex. Suppose \( f: P \to X \) and \( g: Q \to X \) are maps. Let \( X_0 \subset X_1 \subset \cdots \subset X_n = X \) be \( F_\sigma \) subsets of \( X \) obtained from Lemma 2. Then \( f \) can be approximated by \( f': P \to X \) such that \( f'(P) \cap X_i \subset X_i - X_{n-p-1} \), by applying Lemma 3.

Observe that \( f'(P) \cap X_{2n-p-q-1} \) is a \( \sigma \)-compact subset of \( X_{2n-p-q-1} - X_{n-p-1} \), which has dimension \( n - q - 1 \). We wish to approximate \( g \) by \( g': Q \to X \) so that \( g'(Q) \cap f'(P) \cap X_{2n-p-q-1} = \emptyset \), for then it would follow that \( g'(Q) \cap f'(P) \subset X - X_{2n-p-q-1} \), which has dimension \( p + q - n \), as desired.
Recall that \( L_r = \{ x \in \mathbb{R}^m | \text{at most } m-r \text{ coordinates of } x \text{ are rational} \} = \bigcup_{j=1}^{\infty} E_j \), where \( E_1 \subset E_2 \subset \cdots \) and each \( E_j \) is a finite union of \( r \)-dimensional hyperplanes [5]. Thus each \( X_i = \bigcup_{j=1}^{\infty} Y_{ij} \) where \( Y_{ij} \) is a closed subset of \( X \) of dimension \( \leq i \) and \( Y_{ij} \) is 1-LCC in \( X \) if \( i \leq n-3 \). For \( i = 2n-p-q-1 \), \( Y_{ij} \cap f'(P) \) is a compact set of dimension \( \leq n-q-1 \).

Assume first that \( i = 2n-p-q-1 \leq n-3 \). Then \( H_k(X, X -(Y_{ij} \cap f'(P))) = 0 \) for \( 0 \leq k \leq q \) and so \( \pi_k(X, X -(Y_{ij} \cap f'(P))) = 0 \) for \( 0 \leq k \leq q \); hence, we can choose \( g': Q \to X \) approximating \( g \) so that \( g'(Q) \cap (Y_{ij} \cap f'(P)) = \emptyset \) for all \( j \). That is, \( g'(Q) \cap f'(P) \cap X_{2n-p-q-1} = \emptyset \).

Suppose next that \( i = 2n-p-q-1 = n-2 \), \( p \leq q \) (without loss of generality), and \( n \geq 6 \). Then \( n-p \geq 3 \) and we can assume further that \( f'(P) \) and, hence, \( f'(P) \cap Y_{ij} \) are 1-LCC in \( X \). From this we see that \( \pi_k(X, X -(f'(P) \cap Y_{ij})) = 0 \) for \( 0 \leq k \leq q \) and so we can find \( g' \) such that \( g'(Q) \cap f'(P) \cap Y_{ij} = \emptyset \).

Suppose \( 2n-p-q-1 = n-1 \), \( n \geq 5 \), and \( p \leq q \). Then again \( n-p \geq 3 \) and the above argument works.

Finally, if \( 2n-p-q-1 \leq n-2 \) and \( n = 5 \), then the only case in which the above argument does not work is \( p = q = 3 \).

Suppose then that \( n = 5 \) and \( p = q = 3 \). Let \( \{ f_i \}_{i=1}^{\infty} \) be a countable dense subset of \( \text{Map}(D^2, X) \) consisting of 1-LCC embeddings. Given \( f: P \to X \) and \( g: Q \to X \), approximate \( f \) by \( f' \) such that \( \dim(f'(P) \cap f_i(D^2)) \leq 0 \) for all \( i = 1,2,\ldots \) and \( \dim f'(P) \leq 3 \). Then \( f'(P) \) and \( f'(P) \cap f_i(D^2), i = 1,2,\ldots \), are 1-LCC in \( \mathbb{R}^m \) since \( X \) is. Thus for any \( \varepsilon > 0 \) it is possible to find an \( \varepsilon \)-homeomorphism \( h: \mathbb{R}^m \to \mathbb{R}^m \) such that \( h(f'(P) \cap f_i(D^2)) \cap L_{m-2} = \emptyset, \) \( i = 1,2,\ldots \), \( h(X) \cap L_k = \emptyset \), for \( k \leq m-4 \), and \( h(f'(P)) \cap L_{m-2} \) is a 1-dimensional \( \sigma \)-compact set that is 1-LCC in \( X \).

Now replace \( X_i = X \cap L_{m-5+i} \) by \( Y_i = X \cap h^{-1}(L_{m-5+i}) \). Then \( f'(P) \cap Y_3 \) is a 1-dimensional \( \sigma \)-compact set that is 1-LCC in \( X \). Hence, \( g \) can be approximated by \( g': Q \to X \) such that \( g'(Q) \cap f'(P) \cap Y_3 = \emptyset \). This means that \( g'(Q) \cap f'(P) \subset X - Y_3 \), which has dimension 1.

REFERENCES

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