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GENERAL POSITION THEOREMS FOR GENERALIZED MANIFOLDS

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ABSTRACT. It is an open question as to whether a generalized n-manifold, $n \geq 5$, that satisfies the disjoint disks property is a topological manifold. In this paper it is shown that any such space X satisfies general position properties for maps of polyhedra into X.

The program to establish a topological characterization of euclidean space has been stalled, at least temporarily, by the discovery that Quinn's obstruction to resolving a generalized manifold may not be categorically zero as first asserted in [**6**]. (See [**7**].) Thus Cannon's conjecture [**3**] that a generalized n-manifold X, $n \geq 5$, is a topological manifold if and only if X satisfies the disjoint disks property remains open. Quinn's results [**6**, **7**], however, do produce an obstruction $\sigma(X)$ that vanishes if and only if X is a topological manifold, although it is not known at this time whether the obstruction can be nonzero.

In this paper we show (Theorem 3) that any generalized n-manifold X, $n \geq 5$, having the disjoint disks property also satisfies general position properties for maps of arbitrary polyhedra into X. Thus, in a dimension theory sense X behaves like a topological manifold. Theorems of this type have also been obtained by John Walsh [**9**].

DEFINITIONS. A *generalized n-manifold*, n-gm, is a euclidean neighborhood retract X that is also a *homology n-manifold*; that is, $H_*(X, X - x) \cong H_*(\mathbf{R}^n, \mathbf{R}^n - 0)$ for all $x \in X$. (All homology is understood to have integer coefficients.) The space X is said to satisfy the *disjoint k-disks property*, $\mathrm{DD}^k\mathrm{P}$ (or, simply, the disjoint disks property, DDP, when $k = 2$) if for every pair of maps $f_1, f_2 \colon D^k \to X$ of the unit k-disk D^k of $\mathbf{R}^k$ into X and $\varepsilon > 0$, there are maps $f_1', f_2' \colon D^k \to X$ such that $d(f_i', f_i) < \varepsilon$ and $f_1'(D^k) \cap f_2'(D^k) = \varnothing$. If $f, g \colon A \to X$ and $\varepsilon > 0$, then $f \simeq_\varepsilon g$ means f is ε-homotopic to g. A subset A of X is said to be 1-LCC in X (1-locally coconnected in X) if for every $a \in A$ and neighborhood U of a in X there is a neighborhood V of a in X such that the inclusion induced homomorphism $\pi_1(V - A) \to \pi_1(U - A)$ is zero. A map (or embedding) $f \colon A \to X$ is said to be 1-LCC provided $f(A)$ is 1-LCC in X.

Main results. In [**3**] Cannon shows that if X is an n-gm, $n \geq 5$, having the DDP, then an arbitrary map of a 2-disk (hence, a 2-dimensional polyhedron) into

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X can be approximated by 1-LCC embeddings. Our first result is

THEOREM 1. *If X is an n-gm, $n \geq 5$, satisfying the DDP, P is a k-dimensional polyhedron, $2k+1 \leq n$, and $f\colon P \to X$, then f can be approximated by 1-LCC embeddings.*

We show that an n-gm X with the DDP is never "ghastly" [**4**].

THEOREM 2. *Suppose X is an n-gm, $n \geq 5$, having the DDP, P is a k-dimensional polyhedron, and $f\colon P \to X$ is a map. Then f can be approximated by a map $f'\colon P \to X$ such that*
(1) $\dim f'(P) \leq k$, *and*
(2) $f'(P)$ *is 1-LCC if* $k \leq n-3$.

Finally, we obtain the main general position property.

THEOREM 3. *Suppose X is an n-gm, $n \geq 5$, with the DDP, P and Q are polyhedra of dimensions p and q, respectively, and $f\colon P \to X$ and $g\colon Q \to X$ are maps. Then f and g can be approximated by maps f' and g' such that*
(1) $\dim[f'(P) \cap g'(Q)] \leq p+q-n$, *and*
(2) $f'(P) \cap g'(Q)$ *is 1-LCC in X if* $p+q-n \leq n-3$.

Proofs of the main results. We start with a well-known fact (proved, for example, in [**2**, **8**]).

LEMMA 0. *A locally compact, finite-dimensional, separable metric space X admits a 1-LCC embedding in some euclidean space.*

LEMMA 1. *Suppose X is an n-dimensional, 1-LCC closed subset of $\mathbf{R}^m$, $m-n \geq 3$. Then there are F_σ subsets $X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that*
(1) $\dim X_i \leq i$, *and*
(2) $\dim(X_i - X_j) \leq i-j-1$.

PROOF. Let $L_k \subset \mathbf{R}^m$ denote the Menger space:

$$L_k = \{x \in \mathbf{R}^m \mid \text{at least } m-k \text{ coordinates of } x \text{ are rational}\}.$$

Then $\dim L_k = k$, L_k is a countable union of k-dimensional hyperplanes, and $\dim(\mathbf{R}^m - L_k) = m-k-1$ [**5**]. Also $L_0 \subset L_1 \subset \cdots \subset L_m = \mathbf{R}^m$ and $\dim(L_r - L_s) = r-s-1$. Since X is 1-LCC in $\mathbf{R}^m$, we can assume that $L_k \cap X = \varnothing$ for $k \leq m-n-1$ [**2**, **8**].

Set $X_i = X \cap L_{m-n+i}$. Then
(1)
$$\begin{aligned}\dim X_i &= \dim(X \cap L_{m-n+i})\\ &= \dim(X \cap (L_{m-n+i} - L_{m-n-1}))\\ &\leq (m-n+i)-(m-n-1)-1 = i,\end{aligned}$$

and
(2)
$$\begin{aligned}\dim(X_i - X_j) &= \dim(X \cap (L_{m-n+i} - L_{m-n+j}))\\ &\leq \dim(L_{m-n+i} - L_{m-n+j}) = i-j-1.\end{aligned}$$

Throughout the rest of this paper X is an n-gm satisfying the DDP, 1-LCC embedded in $\mathbf{R}^m$ for some $m \geq n+3$.

LEMMA 2. *There are F_σ subsets $X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that*
(1) $\dim X_i \leq i$,
(2) $\dim(X_i - X_j) \leq i - j - 1$, *and*
(3) X_i *is 1-LCC in X for $i \leq n-3$.*

PROOF. Let $\{f_i\}_{i=1}^{\infty}$ be a countable dense subset of $\mathrm{Map}(D^2, X)$, the space of maps (with the compact-open topology), consisting of 1-LCC embeddings [**3**]. In the proof of Lemma 1, require that $L_k \cap f_i(D^2) = \varnothing$ for $k \leq m-3$ and all i (using the fact that the 1-LCC property is transitive). Then $X_i \cap f_j(D^2) = \varnothing$ for $i \leq n-3$. Thus X_i is 1-LCC in X for $i \leq n-3$.

LEMMA 3. *Let $X_0 \subset X_1 \subset \cdots \subset X_n = X$ be F_σ subsets of X satisfying the conclusion of Lemma 2. Then $\pi_k(X, X - X_i) = 0$ for $k < n - i$.*

PROOF. By local duality in X [**1**, **10**], $H_k(X, X - X_i) = 0$ for $k < n - i$. If $i = n-1$ or $n-2$, then $\pi_k(X, X - X_i) = H_k(X, X - X_i) = 0$. If $i \leq n-3$, then X_i is 1-LCC in X so that $\pi_2(X, X - X_i) = 0$. Thus by the Hurewicz isomorphism theorem, $\pi_k(X, X - X_i) = 0$ for $i \leq n-3$ and $k < n - i$. (We may assume that X, and hence $X - X_i$, $i \leq n-3$, are simply connected by passing to the universal cover of X.)

LEMMA 4. *Suppose $f\colon D^k \to X$ is a map and $\varepsilon > 0$. Then $f \simeq_\varepsilon f'$ such that $f'(D^k) \cap X_{n-k-1} = \varnothing$ and f' is 1-LCC, if $k \leq n-3$.*

PROOF. To get $f'(D^k) \cap X_{n-k-1} = \varnothing$, apply Lemma 3 locally, using induction on the skeleta of a fine triangulation of D^k. Similarly, if D is a 1-LCC 2-cell in X, then for a relative neighborhood U of $\mathrm{Int}\, D$, (rel $\mathrm{Bd}\, D$), $\pi_2(U, U - \mathrm{Int}\, D) = 0$ and $H_i(U, U - \mathrm{Int}\, D) = 0$ for $i \leq n-3$. Thus for $r \leq n-3$ any map $g\colon (D^r, \mathrm{Bd}\, D^r) \to (U, U - \mathrm{Int}\, D)$ is homotopic rel $\mathrm{Bd}\, D^r$ to a map $g'\colon D^r \to U - \mathrm{Int}\, D$. Applying this argument inductively on the skeleta of a fine triangulation of D^k, we can get $f \simeq_\varepsilon f'$ such that $f'(D^k) \cap D = \varnothing$ for any 1-LCC 2-cell D in X. Thus we may simultaneously obtain $f \simeq_\varepsilon f'$ so that f' is 1-LCC and $f'(D^k) \cap X_{n-k-1} = \varnothing$.

COROLLARY. *If $f\colon D^k \to X$ and $\varepsilon > 0$, then $f \simeq_\varepsilon f'$ where f' is 1-LCC, if $k \leq n-3$, and* $\dim f'(D^k) \leq k$.

The proof of Theorem 2 follows immediately.

LEMMA 5. *X satisfies DD^kP for $k < n/2$.*

PROOF. Given $f, g\colon D^k \to X$, apply the Corollary to get $f \simeq_\varepsilon f'$ where f' is 1-LCC and $\dim f'(D^k) \leq k$. Then for any open set U of X, $\pi_i(U, U - f'(D^k)) = 0$ for $i \leq n-k-1$. If $k < n/2$, then $k \leq n-k-1$ and, hence, by a now familiar argument, for any $\varepsilon > 0$, $g \simeq_\varepsilon g'$ where $g'(D^k) \cap f'(D^k) = \varnothing$.

The proof of Theorem 1 now follows easily as in [**3**].

PROOF OF THEOREM 3. It is clearly sufficient to prove the theorem when P is a p-simplex and Q is a q-simplex. Suppose $f\colon P \to X$ and $g\colon Q \to X$ are maps. Let $X_0 \subset X_1 \subset \cdots \subset X_n = X$ be F_σ subsets of X obtained from Lemma 2. Then f can be approximated by $f'\colon P \to X$ such that $f'(P) \cap X_i \subset X_i - X_{n-p-1}$, by applying Lemma 3.

Observe that $f'(P) \cap X_{2n-p-q-1}$ is a σ-compact subset of $X_{2n-p-q-1} - X_{n-p-1}$, which has dimension $n-q-1$. We wish to approximate g by $g'\colon Q \to X$ so that $g'(Q) \cap f'(P) \cap X_{2n-p-q-1} = \varnothing$, for then it would follow that $g'(Q) \cap f'(P) \subset X - X_{2n-p-q-1}$, which has dimension $p+q-n$, as desired.

Recall that $L_r = \{x \in \mathbf{R}^m |$ at most $m-r$ coordinates of x are rational$\} = \bigcup_{j=1}^{\infty} E_j$, where $E_1 \subset E_2 \subset \cdots$ and each E_j is a finite union of r-dimensional hyperplanes [**5**]. Thus each $X_i = \bigcup_{j=1}^{\infty} Y_{ij}$ where Y_{ij} is a closed subset of X of dimension $\leq i$ and Y_{ij} is 1-LCC in X if $i \leq n-3$. For $i = 2n-p-q-1$, $Y_{ij} \cap f'(P)$ is a compact set of dimension $\leq n-q-1$.

Assume first that $i = 2n-p-q-1 \leq n-3$. Then $H_k(X, X-(Y_{ij} \cap f'(P))) = 0$ for $0 \leq k \leq q$ and so $\pi_k(X, X-(Y_{ij} \cap f'(P))) = 0$ for $0 \leq k \leq q$; hence, we can choose $g' \colon Q \to X$ approximating g so that $g'(Q) \cap (Y_{ij} \cap f'(P)) = \varnothing$ for all j. That is, $g'(Q) \cap f'(P) \cap X_{2n-p-q-1} = \varnothing$.

Suppose next that $i = 2n-p-q-1 = n-2$, $p \leq q$ (without loss of generality), and $n \geq 6$. Then $n-p \geq 3$ and we can assume further that $f'(P)$ and, hence, $f'(P) \cap Y_{ij}$ are 1-LCC in X. From this we see that $\pi_k(X, X-(f'(P) \cap Y_{ij})) = 0$ for $0 \leq k \leq q$ and so we can find g' such that $g'(Q) \cap f'(P) \cap Y_{ij} = \varnothing$.

Suppose $2n-p-q-1 = n-1$, $n \geq 5$, and $p \leq q$. Then again $n-p \geq 3$ and the above argument works.

Finally, if $2n-p-q-1 = n-2$ and $n = 5$, then the only case in which the above argument does not work is $p = q = 3$.

Suppose then that $n = 5$ and $p = q = 3$. Let $\{f_i\}_{i=1}^{\infty}$ be a countable dense subset of $\mathrm{Map}(D^2, X)$ consisting of 1-LCC embeddings. Given $f \colon P \to X$ and $g \colon Q \to X$, approximate f by f' such that $\dim(f'(P) \cap f_i(D^2)) \leq 0$ for all $i = 1, 2, \ldots$ and $\dim f'(P) \leq 3$. Then $f'(P)$ and $f'(P) \cap f_i(D^2)$, $i = 1, 2, \ldots$, are 1-LCC in $\mathbf{R}^m$ since X is. Thus for any $\varepsilon > 0$ it is possible to find an ε-homeomorphism $h \colon \mathbf{R}^m \to \mathbf{R}^m$ such that $h(f'(P) \cap f_i(D^2)) \cap L_{m-2} = \varnothing$, $i = 1, 2, \ldots$, $h(X) \cap L_k = \varnothing$, for $k \leq m-4$, and $h(f'(P)) \cap L_{m-2}$ is a 1-dimensional σ-compact set that is 1-LCC in X.

Now replace $X_i = X \cap L_{m-5+i}$ by $Y_i = X \cap h^{-1}(L_{m-5+i})$. Then $f'(P) \cap Y_3$ is a 1-dimensional σ-compact set that is 1-LCC in X. Hence, g can be approximated by $g' \colon Q \to X$ such that $g'(Q) \cap f'(P) \cap Y_3 = \varnothing$. This means that $g'(Q) \cap f'(P) \subset X - Y_3$, which has dimension 1.

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