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RELATIONSHIP BETWEEN THE MEET AND JOIN OPERATORS IN THE LATTICE OF GROUP TOPOLOGIES

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ABSTRACT. Let $L(G)$ be the lattice of all topologies on the group G which make G into a topological group. If τ_1 and τ_2 are Hausdorff group topologies and $\tau_1 \vee \tau_2$ is the discrete topology, then $\tau_1 \wedge \tau_2$ is a Hausdorff topology. If τ_1 and τ_2 are locally compact Hausdorff group topologies, then $\tau_1 \vee \tau_2$ is locally compact if and only if $\tau_1 \wedge \tau_2$ is Hausdorff.

If G is a group, the collection of all topologies that make G into a topological group forms a lattice when set inclusion is used as a partial ordering. As is pointed out in [**4**], the intersection of two group topologies need not be a group topology. Thus this lattice of group topologies is not a sublattice of the lattice of all topologies on G. Recently some results concerning the structure of this lattice have been obtained in [**1**, **2**, and **3**]. The purpose of this paper is to further study this lattice and to obtain relationships between the meet and join operators when G is an abelian group. Hence we shall assume as additional hypothesis throughout the paper that G is an abelian group.

THEOREM 1. *If τ_1 and τ_2 are Hausdorff group topologies and $\tau_1 \vee \tau_2$ is the discrete topology, then $\tau_1 \wedge \tau_2$ is a Hausdorff topology.*

PROOF. Endow $G \times G$ with the product topology $\tau_1 \times \tau_2$. The map $m: G \times G \to G$ defined by $m(g_1, g_2) = g_1 g_2$ is a homomorphism and the resulting quotient topology on G is $\tau_1 \wedge \tau_2$. The antidiagonal $\Delta = m^{-1}(e)$ has $\tau_1 \vee \tau_2$ for its inherited topology.

Let $V_1 \times V_2$ be a basic open set in $G \times G$ such that $(V_1 \times V_2) \cap \Delta \neq \varnothing$. Then we have that $H \subset V_1 \cdot V_2$, where $H = \{\bar{e}\}$. Let $z \in H - \{e\}$. We can find $(x, y) \in V_1 \times V_2$ with $z = xy$. Hence, (x, y) is a limit point of Δ in $G \times G$ and since $G_1 \times G_2$ is Hausdorff, $(V_1 \times V_2) \cap \Delta$ is infinite. Therefore, $\tau_1 \vee \tau_2$ cannot be the discrete topology.

COROLLARY 1. *Let τ_1 be a minimal Hausdorff group topology and τ_2 a Hausdorff group topology. If $\tau_1 \vee \tau_2$ is discrete, then τ_2 is discrete.*

Theorem 1 also provides a partial answer to a question asked by Willard [**5**] concerning the intersection topology obtained from two Hausdorff topologies.

COROLLARY 2. *If τ_1 and τ_2 are Hausdorff group topologies and $\tau_1 \vee \tau_2$ is discrete, then $\tau_1 \cap \tau_2$ is a Hausdorff topology.*

THEOREM 2. *Let P be a topological property preserved by finite products and inherited by closed sets. If τ_1 and τ_2 have property P and $\tau_1 \wedge \tau_2$ is Hausdorff, then $\tau_1 \vee \tau_2$ has property P.*

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PROOF. As in Theorem 1 we have $m\colon (G \times G, \tau_1 \times \tau_2) \to (G, \tau_1 \wedge \tau_2)$ a quotient map. Since $\{e\}$ is closed in $(G, \tau_1 \wedge \tau_2)$, we have that the kernel of m, Δ, is closed in $(G \times G, \tau_1 \times \tau_2)$.

COROLLARY 3. *If τ_1 and τ_2 are locally compact Hausdorff group topologies, then $\tau_1 \vee \tau_2$ is locally compact if and only if $\tau_1 \wedge \tau_2$ is Hausdorff.*

REFERENCES

1. B. Clark and V. Schneider, *The normal extensions of subgroup topologies*, Proc. Amer. Math. Soc. **97** (1986), 163–166.
2. M. Lamper, *Complements in the lattice of all topologies of topological groups*, Arch. Math. (Brunensis) **4** (1974), 221–230.
3. D. Remus, *Zur Struktur des Verbandes der Gruppentopologien*, Dissertation, Universität Hannover, 1983.
4. P. Samuel, *Ultrafilters and compactifications of uniform spaces*, Trans. Amer. Math. Soc. **64** (1948), 100–134.
5. S. Willard, *General topology*, Addison-Wesley, Reading, Mass., 1970.

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